SOLUTION OF TRIANGLE
(PHASE–III)

THEORY AND EXERCISE BOOKLET

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♦ THEORY WITH SOLVED EXAMPLES .... 3 – 33

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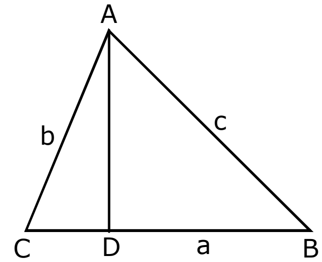
JEE Syllabus :

Relations between sides and angles of a triangle, sine rule, cosine rule, half-angle formula and the area of a triangle.

A. SINE FORMULA

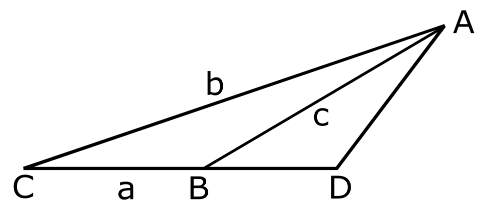
In any triangle the sides are proportional to the sines of the opposite angles i.e.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



(1) Let the triangle ABC be acute-angled.
From A draw AD perpendicular to the opposite side; then
AD = AB sin ($\angle$ABD) = c sin B and AD = AC sin ($\angle$ABCD) = b sin C

$\therefore$ b sin C = c sin B i.e. $\frac{b}{\sin B} = \frac{c}{\sin C}$



(2) Let the triangle ABC have an obtuse angle at B
Draw AD perpendicular to CB produced; then
AD = AC sin $\angle$ACD = b sin C and AD = AB sin $\angle$ABD
= c sin (180º – B) = c sin B;

$\therefore$ b sin C = c sin B i.e. $\frac{b}{\sin B} = \frac{c}{\sin C}$

In a similar manner it may be proved that either of these ratios is equal to $\frac{a}{\sin A}$

Thus $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

Ex.1 If the angles of a ΔABC are $\frac{\pi}{7}$, $\frac{2\pi}{7}$ and $\frac{4\pi}{7}$ and R is the radius of the circumcircle then $a^2 + b^2 + c^2$ has the value equal to

Sol. $a^2 + b^2 + c^2 = 4R^2 (\sin^2 A + \sin^2 B + \sin^2 C) = 2R^2 \left[1 - \cos\frac{2\pi}{7} + 1 - \cos\frac{4\pi}{7} + 1 - \cos\frac{8\pi}{7}\right]$

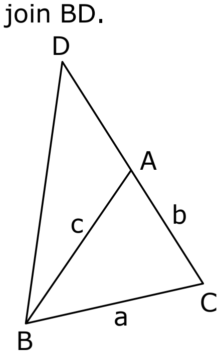
$= 2R^2 [3 - (\cos\theta + \cos 2\theta + \cos 4\theta)]$ where $\theta = 2\pi/7$
now let $S = \cos\theta + \cos 2\theta + \cos 3\theta$ $(\cos 4\theta = \cos 3\theta)$

$2 \sin\frac{\theta}{2} S = \sin\frac{3\theta}{2} - \sin\frac{\theta}{2} + \sin\frac{5\theta}{2} - \sin\frac{3\theta}{2} + \sin\frac{7\theta}{2} - \sin\frac{5\theta}{2} = \sin\frac{7\theta}{2} - \sin\frac{\theta}{2}$

$= \sin\pi - \sin\frac{\theta}{2} = -\sin\frac{\theta}{2}$, $S = -\frac{1}{2}$ $\Rightarrow$ $a^2 + b^2 + c^2 = 2R^2 (3 + 1/2) = 7R^2$

Ex.2 In a triangle ABC, $\angle$A is twice that of show $\angle$B. Whose that $a^2 = b(b + c)$.

Sol. First assume that in the triangle ABC, A = 2B. Produce CA to D such that AD = AB, join BD.
By construction, it is clear that ABD is an isosceles triangle and so $\angle$ADB = $\angle$ABD.
But $\angle$ADB + $\angle$ABD + $\angle$BAC (the external angle)



Hence $\angle$ADB = $\angle$ABD = $\frac{A}{2}$ = B.

In triangles ABC and BDC we have $\angle$ABC = $\angle$BDC and $\angle$C is common. So ΔABC is similar to ΔBDC. Therefore $\frac{AC}{BC} = \frac{BC}{DC}$

If follows that $a^2 = b(b + c)$

Now we will prove the converse. Assume that $a^2 = b(b + c)$. We refer to the same figure. As before, in the isosceles triangle ABD, we have $\angle ABD = \angle ADB$. So each of these angles is equal to half of their sum which is A. Thus, in particular, $\angle ADB = \frac{A}{2}$ ....(1)

On the other hand, in triangles ACB and BCD, we have, as a consequence of the assumption $a^2 = b(b + c)$, $\frac{AC}{BC} = \frac{BC}{DC}$, and $\angle C$ is common. So the two triangles are similar and $\angle CDB = \angle CBA = B$. ....(2)

From (1) and (2), it follows that B = A/2, as desired.

Aliter : We may use the Sine rule for a triangle to dispose of both the implications simultaneously.

$A = 2B \Leftrightarrow A - B = B \Leftrightarrow \sin(A - B) = \sin B \Leftrightarrow \sin(A - B)\sin(A + B) = \sin B \sin C$

$\Leftrightarrow \sin^2 A - \sin^2 B = \sin B \sin C \Leftrightarrow (2R \sin A)^2 - (2R \sin B)^2 = (2R \sin B)(2R \sin C)$

$\Leftrightarrow a^2 - b^2 = bc \Leftrightarrow a^2 = b(b + c)$

Ex.3 In a triangle ABC, a cos A + b cos B + c cos C = s. Prove that the triangle is equilateral.

Sol. The given result can be written as 2a cos A + 2b cos B + 2c cos C = a + b + c

Using sine rule we get 2 sin A cos A + 2 sin B cos B + 2 sin C cos C = sin A + sin B + sin C

$\Rightarrow \sin 2A + \sin 2B + \sin 2C = \sin A + \sin B + \sin C \Rightarrow 4 \sin A \sin B \sin C = 4 \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

$\Rightarrow 8 \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 1 \Rightarrow 4\left[\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right]\sin\frac{C}{2} = 1$

$\Rightarrow 4\sin^2\frac{C}{2} - 4\cos\frac{A-B}{2}\sin\frac{C}{2} + 1 = 0$. This is a quadratic equation in $\sin\frac{C}{2}$ which must have real roots.

Hence $16\cos^2\frac{A-B}{2} \le 1 \Rightarrow \cos^2\frac{A-B}{2} \le 1$. But $\cos^2\frac{A-B}{2} \le 1 \Rightarrow \cos^2\frac{A-B}{2} = 1 \Rightarrow A = B$

Similarly it can be prove that $B = C \Rightarrow A = B = C$

B. COSINE FORMULA

To find an expression for one side (c) of a triangle in terms of other to sides and the included angle (C).

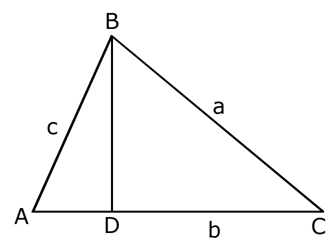
(1) Let C be an acute angle.

Draw BD perpendicular to AC ;

$AB^2 = BC^2 + CA^2 - 2AC \cdot CD$;

$\therefore \quad c^2 = a^2 + b^2 - 2ba \cos C$

$= a^2 + b^2 - 2ab \cos C.$



(2) Let C be an obtuse angle.

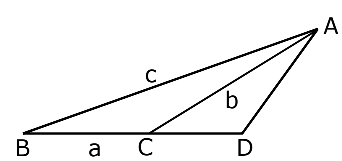
Draw BD perpendicular to AC produced ;

$AB^2 = BC^2 + CA^2 + 2AC \cdot CD$;

$\therefore \quad c^2 = a^2 + b^2 + 2ba \cos BCD$

$= a^2 + b^2 + 2ab \cos(180° - C)$

$= a^2 + b^2 - 2ab \cos C$



Hence in each case, $c^2 = a^2 + b^2 - 2ab \cos C$

Similarly it may be shown that

$a^2 = b^2 + c^2 - 2bc \cos A$ and $b^2 = c^2 + a^2 - 2ac \cos B$

From the above formulae we obtain

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \; ; \cos B = \frac{c^2 + a^2 - b^2}{2ca} \; ; \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

These results enable us to find the cosines of the angles when the numerical values of the sides are given.

Ex.4 If the sides a, b, c of a ΔABC satisfy the relation, $a^4 + b^4 + c^4 = 2c^2 (a^2 + b^2)$, find the possible values of the angle C.

Sol. Solving as a quadratic equation in c we get, $c^2 = a^2 + b^2 \pm \sqrt{2}\,ab$

or $\quad a^2 + b^2 - c^2 = \pm \sqrt{2}\,ab = \dfrac{a^2 + b^2 - c^2}{2ab} = \pm \dfrac{1}{\sqrt{2}} \quad \Rightarrow \quad C = \dfrac{\pi}{4}$ or $\dfrac{3\pi}{4}$

Ex.5 Let a, b, c be the sides of a triangle and Δ is its area. Prove that $a^2 + b^2 + c^2 \geq 4\sqrt{2}\;\Delta$. When does the equality hold ?

Sol. TPT : $a^2 + b^2 + (a^2 + b^2 - 2ab \cos C) \geq 4\sqrt{3}\;\dfrac{1}{2} \sin C$ or $\dfrac{a}{b} + \dfrac{b}{a} \geq \sqrt{3} \sin C + \cos C$

$$\Rightarrow \underbrace{\left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)^2 + 2}_{\text{Min. value 2}} \geq \underbrace{2\sin\left(C + \frac{\pi}{6}\right)}_{\text{Min. value 2}}$$

. Equality occurs when a = b and $C = \pi/3$

Ex.6 In any ΔABC, prove that $(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0$

Sol. $\because$ from sine rule, we know that a = k sin A, b = k sin B and c = k sin C

$\therefore$ $(b^2 - c^2) \cot A = k^2 (\sin^2 B - \sin^2 C) \cot A \qquad \because \quad \sin^2 B - \sin^2 C = \sin (B + C) \sin (B - C)$

$\therefore$ $(b^2 - c^2) \cot A = k^2 \sin (B + C) \sin (B - C) \cot A \qquad \because \quad B + C = \pi - A$

$\therefore$ $(b^2 - c^2) \cot A = k^2 \sin A \sin (B - C) \dfrac{\cos A}{\sin A} \qquad \because \quad \cos A = -\cos (B + C)$

$$= -k^2 \sin (B - C) \cos (B + C) = -\frac{k^2}{2} [2 \sin (B - C) \cos (B + C)]$$

$$\Rightarrow (b^2 - c^2) \cot A = -\frac{k^2}{2} [\sin 2B - \sin 2C] \qquad \text{....(i)}$$

Similarly $(c^2 - a^2) \cot B = -\dfrac{k^2}{2} [\sin 2C - \sin 2A]$ ....(ii)

and $(a^2 - b^2) \cot C = -\dfrac{k^2}{2} [\sin 2A - \sin 2B]$ ....(iii)

adding equation (i), (ii) and (iii), we get $(b^2 - c^2) \cot A + (c^2 - b^2) \cot B + (a^2 - b^2) \cot C = 0$

Ex.7 In a ΔABC, prove that $a\,(b \cos C - c \cos B) = b^2 - c^2$

Sol. $\because$ We have to prove $a\,(b \cos C - c \cos B) = b^2 - c^2$

$\because$ from cosine rule we know that $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$ & $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

$$\therefore \text{ L.H.S.} = a\left\{b\left(\frac{a^2 + b^2 - c^2}{2ab}\right) - c\left(\frac{a^2 + c^2 - b^2}{2ac}\right)\right\} = \frac{a^2 + b^2 - c^2}{2} - \frac{(a^2 + c^2 - b^2)}{2} = (b^2 - c^2) = \text{R.H.S.}$$

Ex.8 If in ΔABC, $\angle A = 60°$ then find the value of $\left(1 + \dfrac{a}{c} + \dfrac{b}{c}\right)\left(1 + \dfrac{c}{b} - \dfrac{a}{b}\right)$.

Sol. $\because$ $\angle A = 60°$ $\quad\because$ $\left(1 + \dfrac{a}{c} + \dfrac{b}{c}\right)\left(1 + \dfrac{c}{b} - \dfrac{a}{b}\right) = \left(\dfrac{c + a + b}{c}\right)\left(\dfrac{b + c - a}{b}\right)$

$$= \frac{(b + c)^2 - a^2}{bc} = \frac{(b^2 + c^2 - a^2) + 2bc}{bc} = \frac{b^2 + c^2 - a^2}{bc} + 2 = 2\left(\frac{b^2 + c^2 - a^2}{2bc}\right) + 2$$

$= 2 \cos A + 2 = 3$ $\quad\left(\because\ \angle A = 60° \Rightarrow \cos A = \dfrac{1}{2}\right)$ $\quad\therefore$ $\left(1 + \dfrac{a}{c} + \dfrac{b}{c}\right)\left(1 + \dfrac{c}{b} - \dfrac{a}{b}\right) = 3$

C. PROJECTION FORMULA

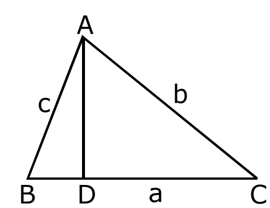
To express one side of a triangle in terms of the adjacent angles and the other two sides.

(1) Let ABC be an acute-angled triangle

Draw AD perpendicular to BC;

then $BC = BD + CD = AB \cos \angle ABD + AC \cos \angle ACD$;

i.e. $a = c \cos B + \cos C$



(2) Let the triangle ABC have an obtuse angle C.

Draw AD perpendicular to BC produced; then

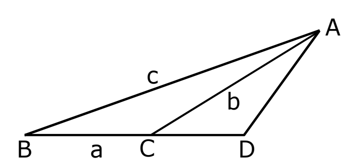
$BC = BD - CD = AB \cos \angle ABD - AC \cos \angle ACD$;

$\therefore$ $a = \cos B - b \cos (180° - C) = c \cos B + b \cos C$

Thus in each each case $a = b \cos C + c \cos B$.

Similarly it may be shown that

$b = c \cos A + a \cos C$, and $c = a \cos B + b \cos A$



Ex.9 In a ΔABC prove that $(b + c) \cos A + (c + a) \cos B + (a + b) \cos C = a + b + c$

Sol. $\because$ L.H.S. $= (b + c) \cos A + (c + a) \cos B + (a + B) \cos C$

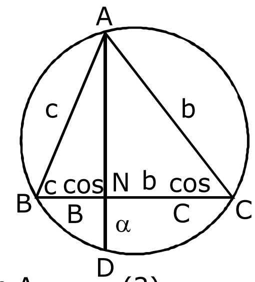
$= b \cos A + c \cos A + c \cos B + a \cos B + a \cos C + b \cos C$

$= (b \cos A + a \cos B) + (c \cos A + a \cos C) + (c \cos B + b \cos C) = a + b + c =$ R.H.S.

Ex.10 In a ΔABC perpendiculars are drawn from angles A, B, C of an acute angled triangle on the opposite sides and produced to meet the circumscribing circle. If these produced points be α, β, γ respectively, show that $\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 2\,\Pi \tan A$, where Π denotes the continued product.

Sol. Using the property of cyclic quadrilateral,

c sin B . α = bc cos B . cos C



$$\frac{\alpha}{a} = \frac{b}{a}\,\frac{\cos B.\cos C}{\sin B} = \frac{\sin B}{\sin A}\,.\,\frac{\cos B.\cos C}{\sin B} = \frac{\cos B.\cos C}{\sin A}$$

$$\frac{a}{\alpha} = \frac{\sin A}{\sin B\cos C} = \frac{\sin(B+C)}{\cos B\cos C} = \frac{\sin B\cos C + \cos B\sin C}{\cos B\cos C}$$

$\frac{a}{\alpha}$ = tan B + tanC ....(1) similarly $\frac{b}{\beta}$ = tan C + tan A ....(2)

and $\frac{c}{\gamma}$ = tan A + tan B ....(3) adding $\Sigma \frac{a}{\alpha} = 2\,\Sigma \tan A = 2\,\Pi \tan A$

Ex.11 In any triangle ABC, $(a + b)^2 \sin^2 \frac{C}{2} + (a - b)^2 \cos^2 \frac{C}{2}$ =

Sol. $a^2\left(\sin^2\frac{C}{2}+\cos^2\frac{C}{2}\right) + b^2\left(\sin^2\frac{C}{2}+\cos^2\frac{C}{2}\right) - 2ab\left(\cos^2\frac{C}{2}-\sin^2\frac{C}{2}\right) = a^2 + b^2 - 2ab\cos C = c^2$

D. NAPIER'S ANALOGY - TANGENT RULE

(i) $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$ **(ii)** $\tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot\frac{B}{2}$ **(iii)** $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$

Ex.12 Find the unknown elements of the ΔABC in which $a = \sqrt{3} + 1$, $b = \sqrt{3} - 1$, C = 60º

Sol. $\because$ $a = \sqrt{3} + 1$, $b = \sqrt{3} - 1$, C = 60º $\because$ A + B + C = 180º $\therefore$ A + B = 120º ....(i)

$\because$ From law of tangent, we know that $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2} = \frac{(\sqrt{3}+1)-(\sqrt{3}-1)}{(\sqrt{3}+1)+(\sqrt{3}-1)}\cot 30^\circ$

$= \frac{2}{2\sqrt{3}}\cot 30^\circ \Rightarrow \tan\left(\frac{A-B}{2}\right) = 1 \quad \therefore \frac{A-B}{2} = \frac{\pi}{4} = 45^\circ \Rightarrow A - B = 90^\circ$ ....(ii)

From equation (i) and (ii); we get A = 105º and B = 15º. Now,

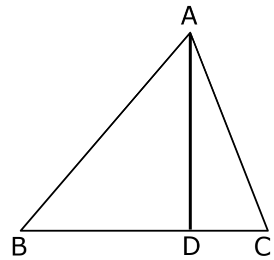
$\because$ From sine-rule, we know that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$\therefore$ $c = \frac{a\sin C}{\sin A} = \frac{(\sqrt{3}+1)\sin 60^\circ}{\sin 105^\circ} = \frac{(\sqrt{3}+1)\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}+1}{2\sqrt{2}}}$ $\because$ $\sin 105^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$

$\Rightarrow$ $c = \sqrt{6}$ $\therefore$ $c = \sqrt{6}$, A = 105º, B = 15º

E. AREA OF A TRIANGLE

To find the area of a triangle. Let Δ denote the area of the triangle ABC.Draw AD perpendicular to BC. The area of a triangle is half the area of a rectangle on the same base and of the same altitude



$\therefore \quad \Delta = \frac{1}{2}$ (base × altitude)

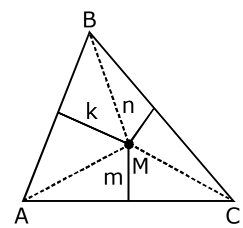
$= \frac{1}{2}BC \,.\, AD = \frac{1}{2}\,BC \,.AB \sin B = \frac{1}{2}\,ca \sin B$

$\because\ R = \frac{abc}{4\Delta} \Rightarrow abc = 4R\Delta \ \therefore\ LHS = \frac{4R}{\Delta}$

Ex.13 Find the area of a triangle with angles α, β and γ knowing that the distances from an arbitrary point M taken inside the triangle to its sides are equal to m,nand k. (fig.)

Sol. The area S of the triangle ABC can be found by the formula S = $\frac{1}{2}$ AC. BC. sin γ, for which purpose we have to find AC and BC. Let BC = x. Then, by the law of sines, we have $\frac{AC}{\sin\beta} = \frac{BC}{\sin\alpha} = \frac{AB}{\sin\gamma}$,

whence we find that AC = $\frac{x\sin\beta}{\sin\alpha}$ and AB = $\frac{x\sin\gamma}{\sin\alpha}$.



Thus, the problem is reduced to finding x. To set up an equation, we are going to apply the method of areas, taking the area S of the triangle ABC as a reference element. We have :

$S = \frac{1}{2}\,AC \,.\, BC \,.\, \sin\gamma = \frac{1}{2}\,\frac{x\sin\beta}{\sin\alpha}\,.\, x \sin\gamma = \frac{x^2\sin\beta\sin\gamma}{2\sin\alpha}$. On the other hand,

$S = S_{AMB} + S_{BMC} + S_{AMC} = \frac{1}{2}AB \,.\, k + \frac{1}{2}BC \,.\, n + \frac{1}{2}AC \,.\, m = \frac{1}{2}\,.\,\frac{x\sin\gamma}{\sin\alpha}\,.\, k + \frac{1}{2}xn + \frac{1}{2}\,.\,\frac{\sin\beta}{\sin\beta}\,.m$

$= \frac{x(k\sin\gamma + n\sin\alpha + m\sin\beta)}{2\sin\alpha}$

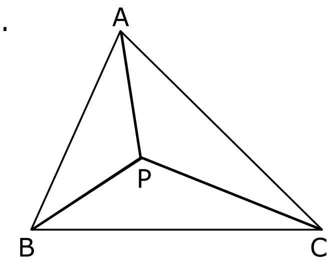
Hence, $\frac{x^2\sin\beta\sin\gamma}{2\sin\alpha} = \frac{x(k\sin\gamma + \sin\alpha + m\sin\beta)}{2\sin\alpha}$, whence we get : $x = \frac{x^2\sin\gamma + n\sin\alpha + m\sin\beta}{\sin\beta\sin\gamma}$.

Substituting this value of x into the first of the above formulas for the area of the triangle ABC, we obtain : $S = \frac{x^2\sin\beta\sin\gamma}{2\sin\alpha} = \frac{(k\sin\gamma + n\sin\alpha + m\sin\beta)^2}{2\sin\alpha\sin\beta\sin\gamma}$.

Ex.14 Let P be a point inside the triangle ABC such that $\angle APB = \angle BPC = \angle CPA$. Prove that

$PA + PB + PC = \sqrt{\frac{a^2+b^2+c^2}{2} + 2\sqrt{3}\Delta}$, where a, b, c, Δ are the sides and the area of triangle ABC.

Sol. Since $\angle APB = \angle BPC = \angle CPA$ $\Rightarrow$ Each of these angle is equal to $\frac{2\pi}{3}$.



In triangle APC we have $PA^2 + PC^2 - 2PA \,.\, PC \cos\frac{2\pi}{3} = b^2$

$\Rightarrow$ $b^2 = PA^2 + PC^2 + PA \,.\, PC$

Similarly in ΔBPC; $a^2\ PB^2 + PC^2 + PB \,.\, PC$ and

in ΔBPC ; $c^2 = AP^2 + PB^2 + PA \,.\, PB$

Adding these results we get $a^2 + b^2 + c^2 = 2(PA^2 + PB^2 + PC^2) + PA \,.\, PB + PB \,.\, PC + PC \,.\, PA$

$\Rightarrow$ $a^2 + b^2 + c^2 = 2[(PA + PB + PC))^2 - 2\Sigma PA \,.\, PB)] + \Sigma PA \,.\, PB = 2(PA + PB + PC)^2 - 3\Sigma PA \,.\, PB$

Now, $\Delta = \Delta_{APC} + \Delta_{BPC} + \Delta_{BPA} = \frac{1}{2}\sin\frac{2\pi}{3}(AP \,.\, PC + BP \,.\, PC + PA \,.\, PB) = \frac{\sqrt{3}}{4}(\Sigma PA \,.\, PB)$

Putting back the value of $\Sigma PA \,.\, PB$ we get ; $a^2 + b^2 + c^2 = 2(AP + PB + PC)^2 - 3 \,.\, \frac{4\Delta}{\sqrt{3}}$

$\Rightarrow$ $(PA + PB + PC)^2 = \frac{a^2+b^2+c^2}{2} + 2\sqrt{3}\,\Delta \Rightarrow PA + PB + PC = \sqrt{\frac{a^2+b^2+c^2}{2} + 2\sqrt{3}\Delta}$.

F. TRIGONOMETRIC FUNCTIONS OF HALF ANGLES

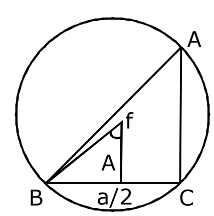
(i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$; $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$; $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

(ii) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$; $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$; $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(iii) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{\Delta}{s(s-a)}$ where $s = \frac{a+b+c}{2}$ & Δ = area of triangle

(iv) Area of triangle = $\sqrt{s(s-a)(s-b)(s-c)}$

Ex.15 If a, b, c are in A.P., then the numerical value of $\tan\frac{A}{2}\tan\frac{C}{2}$ is



Sol. Given $2b = a + c$ $\Rightarrow$ $3b = 2s = a + b + c$

$\tan\frac{A}{2}\tan\frac{C}{2} = \frac{\Delta}{s(s-a)} \cdot \frac{\Delta}{s(s-c)} \cdot \frac{s-b}{s-b} = \frac{2s-2b}{2s} = \frac{b}{3b} = \frac{1}{3}$

Ex.16 In a ΔABC, if cos A + cos B = 4 $\sin^2 \frac{C}{2}$, prove that $\tan \frac{A}{2} \cdot \tan \frac{B}{2} = \frac{1}{3}$. Hence deduce that the sides of the triangle are in A.P.

Sol. $2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} = 4 \sin^2 \frac{C}{2}$ or $\cos \frac{A-B}{2} = 2 \sin \frac{C}{2} \left(\sin\frac{C}{2} = \cos\frac{A+B}{2}\right)$

$= 2 \cos \frac{A+B}{2}$ or $\cos \frac{A-B}{2} - \cos \frac{A+B}{2} = \cos \frac{A+B}{2}$

$2 \sin \frac{A}{2} \cdot \sin \frac{B}{2} = \cos \frac{A}{2} \cdot \cos \frac{B}{2} - \sin \frac{A}{2} \cdot \sin \frac{B}{2}$

$3 \sin \frac{A}{2} \cdot \sin \frac{B}{2} = \cos \frac{A}{2} \cdot \cos \frac{B}{2}$ or $\tan \frac{A}{2} \cdot \tan \frac{B}{2} = \frac{1}{3}$

Now $\frac{\Delta}{s(s-a)} \cdot \frac{\Delta}{s(s-b)} = \frac{1}{3} \quad \Rightarrow \quad \frac{s-c}{s} = \frac{1}{3}$

$\therefore$ 2s = 3c $\Rightarrow$ a + b + c = 3c $\Rightarrow$ a + b = 2c $\Rightarrow$ a, c, b are in A.P.

Ex.17 With usual notions, prove that in a triangle ABC, $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \frac{s}{r}$.

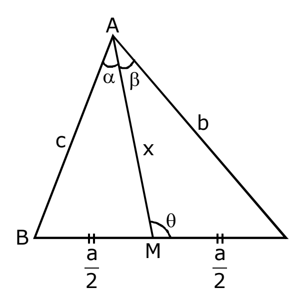
Sol. Using $\cot \frac{A}{2} = \frac{s(s-a)}{\Delta}$ etc.

$$\text{LHS} = \frac{s(s-a)+s(s-b)+s(s-c)}{\Delta} = \frac{3s^2 - s(a+b+c)}{\Delta} = \frac{3s^2 - 2s^2}{\Delta} = \frac{s^2}{\Delta} = \frac{s}{r}$$

G. M-N RULE

In any triangle, (m + n) cot θ = m cot α – n cot β = n cot B – m cot C

Ex.18 Prove that the median through A divides angle A into two parts whose cotangents are, 2 cot A + cot C and 2 cot A + cot B makes on angle with the side BC whose cotangent is $\frac{1}{2}$ (cot B – cot C).



Sol. In ΔAMC, using cosine rule $x^2 = b^2 + \frac{a^2}{4} - ab \cos C$

Substituting $\cos C = \frac{a^2+b^2-c^2}{2ab}$ & simplifying $4x^2 = 2b^2 + 2c^2 - a^2$

Now using m – n theorem 2 cot θ = cot α – cot β ....(1)
2 cot θ = cot B – cot C ....(2)
Hence cot α – cot β = cot B – cot C ....(3)

Now $\frac{x}{\sin B} = \frac{a}{2\sin\alpha}$ $\Rightarrow$ $\sin\alpha = \frac{a\sin B}{2x}$ and $\cos\alpha = \frac{c^2 + x^2 - \frac{a^2}{4}}{4cx}$ $\Rightarrow$ $2cx\cos\alpha = \frac{4c^2 + 4x^2 - a}{4}$

and $\cos\alpha = \frac{3c^2 + b^2 - a^2}{4cx}$ (on putting the value of $4x^2$ from above)

Hence $\cot\alpha = \frac{3c^2 + b^2 - a^2}{2ac\sin B} = \frac{3c^2 + b^2 - a^2}{4\Delta} = \cot\alpha$ ....(4)

Similarly $\cot\beta = \frac{3b^2 + c^2 - a^2}{4\Delta}$ ....(5)

Hence $\cot\alpha + \cot\beta = \frac{4c^2 + 4b^2 - 2a^2}{4\Delta} = \frac{4(c^2 + b^2 - a^2) + 2a^2}{4\Delta} = \frac{4(c^2 + b^2 - a^2) + (a^2 + b^2 - c^2) + c^2 + a^2 - b^2}{4\Delta}$

Hence $\cot\alpha + \cot\beta = 4\cot A + \cot C + \cot B$ ....(6)

add (3) & (6) and subtract (3) and (6) we get the result

Alternative solution : $\frac{x}{\sin B} = \frac{a}{2\sin\alpha}$ and $\frac{x}{\sin C} = \frac{a}{2\sin\beta}$

On dividing $\frac{\sin C}{\sin B} = \frac{\sin\beta}{\sin\alpha} = \frac{\sin(A-\alpha)}{\sin\alpha} \Rightarrow \frac{\sin(A+B)}{\sin B} = \frac{\sin(A-\alpha)}{\sin\alpha}$

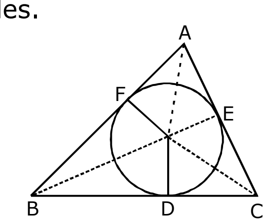
$\sin A\cot B + \cos A = \sin A\cot\alpha - \cos A$ or $\cot B + \cot A = \cot\alpha - \cot A$ or $2\cot A + \cot B = \cot\alpha$

H. RADIUS OF THE INCIRCLE

To find the radius of the circle inscribed in a triangle. Let ℓ be the circle inscribed in the triangle ABC, and D, E, F the points of contact; then ID, IE, IF are perpendicular to the sides.

Now Δ = sum of the areas of the triangles BIC, CIA, AIB

$= \frac{1}{2}$ ar + $\frac{1}{2}$ br + $\frac{1}{2}$ or $= \frac{1}{2}(a + b + c)r = sr \Rightarrow r = \frac{\Delta}{s}$



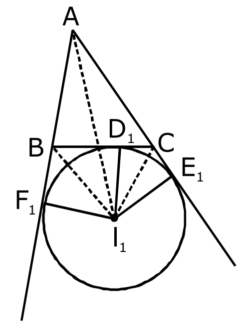
(a) $r = \frac{\Delta}{s}$ where $s = \frac{a+b+c}{2}$ **(b)** $r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$

(c) $r = \frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}$ & so on **(d)** $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

I. RADIUS OF THE EX-CIRCLES

A circle which touches one side of a triangle and the other two sides produced is said to be an escribed circle of the triangle. Thus the triangle ABC has three escribed circles, one touching BC, and AB, AC produced, a second touching CA, and BC produced ; a third touching AB, and CA, CB produced.

To find the radius of an escribed circle of a triangle. Let I_1 be the centre of the circle touching the side BC and the two sides AB and AC produced. Let D_1, E_1, F_1 be the points of contact; then the lines joining I_1 to these points are perependicular to the sides.



Let r_1 be the radius ; then

Δ = area ABC = area ABI_1C – area BI_1C = area BI_1A + area CI_1A – area BI_1C

$$= \frac{1}{2}cr_1 + \frac{1}{2}br_1 - \frac{1}{2}ar_1 = \frac{1}{2}(c + b + a)\, r_1 = (s - a)\, r_1 \quad \therefore \quad r_1 = \frac{\Delta}{s-a}$$

Similarly, if r_2, r_3 be the radii of the escribed circles opposite to the angles B and C respectively $r_2 = \frac{\Delta}{s-b}$, $r_3 = \frac{\Delta}{s-c}$.

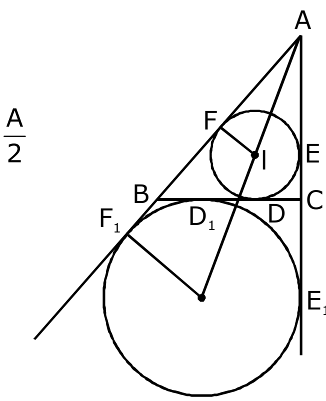
Many important relations connecting a triangle and its circles may be established by elementary geometry.

With the notation of previous articles, since tangents to a circle from the same point are equal.

we have, AF = AE, BD = BF, CD = CE;

$\therefore$ AF + (BD + CD) = half the sum of the sides ;

$\therefore$ AF + a = s $\quad\therefore\quad$ AF = s – a = AE



Similarly, BD = BF = s – b, CD = CE = s – c. Also $r = AF\tan\frac{A}{2} = (s-a)\tan\frac{A}{2}$

Similarly, $r = (s - b)\tan\frac{B}{2}$, $r = (s - c)\tan\frac{C}{2}$.

Again, $AF_1 = AE_1$, $BF_1 = BD_1$, $CE_1 = CD_1$

$\therefore$ $2AF_1 = AF_1 + AE_1 = (AB + BD_1) + (AC + CD_1)$ = sum of the sides

$\therefore$ $AF_1 = s = AE_1$ $\quad\therefore\quad$ $BD_1 = BF_1 = s - c$, $CD_1 = CE_1 = s - b$

Also $r_1 = AF_1\tan\frac{A}{2} = s\tan\frac{A}{2}$ Similarly, $r_2 = s\tan\frac{B}{2}$, $r_3 = s\tan\frac{C}{2}$.

$$\text{Note : } r_1 = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}} = 4R\sin\frac{A}{2}\,.\cos\frac{B}{2}\,.\cos\frac{C}{2} \text{ \& so on}$$

Ex.19 With usual notation in a triangle ABC, prove that $r^2 + s^2 + 4Rr = ab + bc + ca$.

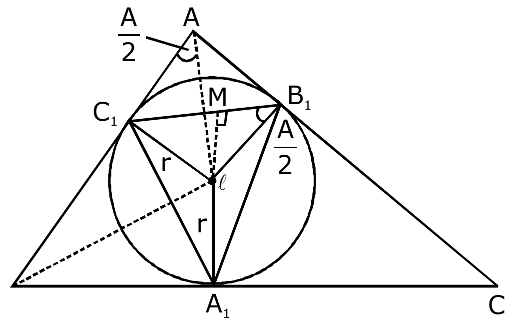
Sol. $$r^2 + s^2 + 4Rr = \frac{\Delta^2}{s^2} + s^2 + \frac{abc}{\Delta}\,.\,\frac{\Delta}{s} \quad \left(r = \frac{\Delta}{s}, R = \frac{abc}{4\Delta}\right)$$

$$= \frac{s(s-a)(s-b)(s-c)}{s.s} + s^2 + \frac{abc}{s} = \frac{(s-a)(s-b)(s-c) + s^2 + abc}{s}$$

$$= \frac{2s^3 - (a+b+c)s^2 + (ab+bc+ca)s - abc + abc}{s} = ab + bc + ca \text{ (proved)}$$

Ex.20 Let the in-circle of the ΔABC touches its sides BC, CA & AB at A_1, B_1 & C_1 respectively. If ρ_1, ρ_2 & ρ_3 are the circum radii of the triangles, B_1IC_1, C_1IA_1 and A_1IB_1 respectively, then prove that, $2\rho_1\rho_2\rho_3 = R\rho^2$ where R is the circumradius and r is the inradius of the ΔABC.

Sol. $AC_1\, IB$ is a cyclic quadrilateral. Hence in ΔB_1C_1I,



$$\frac{B_1C_1}{\sin(\pi - A)} = 2\rho_1 \Rightarrow B_1C_1 = 2\rho_1 \sin A \Rightarrow \rho_1 = \frac{B_1C_1}{2\sin A}$$

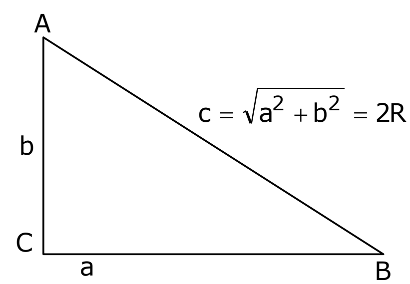
Now in Δ IMB_1 $\cos\dfrac{A}{2} = \dfrac{B_1M}{r} \Rightarrow B_1M = r\cos\dfrac{A}{2}$

$\therefore$ $B_1M = r\cos\dfrac{A}{2}$

Hence $B_1C_1 = 2B_1M = 2r\cos\dfrac{A}{2}$ $\qquad \therefore \quad \rho_1 = \dfrac{2r\cos\frac{A}{2}}{2\sin A} = \dfrac{r}{2\sin\frac{A}{2}}$

$$\therefore \quad 2\rho_1\rho_2\rho_3 = \frac{r^3}{4\pi\sin\frac{A}{2}} = \frac{r^3R}{r} \quad \left(\therefore\ 4\pi\sin\frac{A}{2} = \frac{r}{R}\right) = r^2R$$

Ex.21 Find the area of a right triangle if it is known that the radius of the circle inscribed in the triangle is r and that of the circumscribed circle is R.



Sol. $r = \dfrac{2\Delta}{2S} = \dfrac{2\Delta}{a+b+c} = \dfrac{2\Delta}{a+b+2R}$

$r(a + b) = 2D - 2Rr$

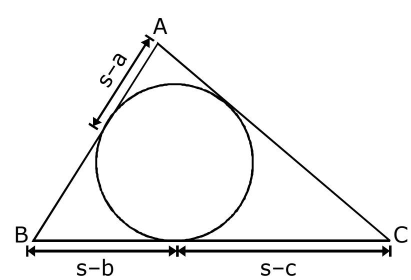
$r^2[a^2 + b^2\ 2ab] = 4(D - Rr)^2$

$r^2[4R^2 + 4D] = 4(D - Rr)^2$ [$\therefore$ $ab = 2D$]

$r^2[R^2 + D^2] = D^2 + R^2r^2 - 2Dr$

$Dr^2 = D^2 - 2Dr \Rightarrow D = r^2 + 2Rr = r(r + 2R)$

Ex.22 If α, β, γ be the distances of the angular points of a triangle from the points of contact of the incircle with the sides of the triangle, then show that $r = \sqrt{\dfrac{\alpha\beta\gamma}{\alpha+\beta+\gamma}}$



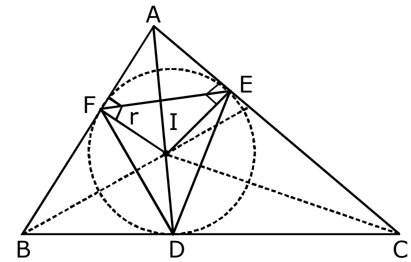
Sol. $\left.\begin{array}{l}\alpha = s - a\\ \beta = s - b\\ \gamma = s - c\end{array}\right] \Rightarrow \quad \alpha + \beta + \gamma = s$

Now $r^2 = \dfrac{\Delta^2}{s^2} = \dfrac{s(s-a)(s-b)(s-c)}{s^2} = \dfrac{(s-a)(s-b)(s-c)}{s} = \dfrac{\alpha\beta\gamma}{\alpha+\beta+\gamma}$. Hence $r = \sqrt{\dfrac{\alpha\beta\gamma}{\alpha+\beta+\gamma}}$

Ex.23 ΔDEF is formed by joining the points of contact of the circle with the sides of the ΔABC. The sides, angles and the area of the ΔDEF are respectively.

Sol. AFIE is a cyclic quadrilateral. Similarly DIEC and DIFB are concyclic.

$\angle FAI = \angle FEI = \dfrac{A}{2}$ and $\angle IED = \angle ICD = \dfrac{C}{2}$ $\Rightarrow$ $\angle FED = \dfrac{A+C}{2} = \left(\dfrac{\pi}{2} - \dfrac{B}{2}\right)$



$\Rightarrow$ angles of ΔDEF are $\left(\dfrac{\pi}{2} - \dfrac{A}{2}\right)$; $\left(\dfrac{\pi}{2} - \dfrac{B}{2}\right)$ and $\left(\dfrac{\pi}{2} - \dfrac{C}{2}\right)$

Now $\dfrac{FE}{\sin(\pi - A)} = AI = \dfrac{r}{\sin\dfrac{A}{2}}$ $\Rightarrow$ $FE = 2r\cos\dfrac{A}{2}$ etc.

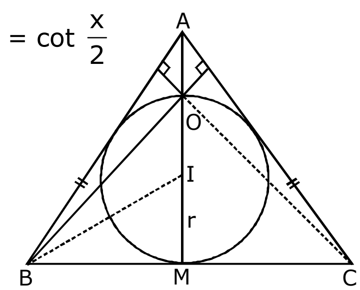
Area of $\Delta DEF = \dfrac{1}{2} \cdot 2r\cos\dfrac{B}{2} \cdot 2r\cos\dfrac{C}{2}\sin\left(\dfrac{\pi}{2} - \dfrac{A}{2}\right) = 2r^2\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

$$= \frac{1}{2}r^2 . (\sin A + \sin B + \sin C) = \frac{1}{2}r^2\left(\frac{a+b+c}{2R}\right) = \frac{r^2 s}{2R} \text{ or}$$

$\Rightarrow$ $\left(2r\cos\dfrac{A}{2}, 2r\cos\dfrac{B}{2}, 2r\cos\dfrac{C}{2}\right)$; $\left(\dfrac{\pi - A}{2}, \dfrac{\pi - B}{2}, \dfrac{\pi - C}{2}\right)$ & $\dfrac{r^2 s}{2R}$ or $2r^2\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

Ex.24 In an isosceles ΔABC if the altitudes intersect on the inscribed circle then the cosine of the vertical angle 'A' is

Sol. $\angle A = x$ $\Rightarrow$ $\angle BOI = \dfrac{\pi}{2} - \dfrac{x}{2}$ and $\angle BIM = \dfrac{\pi}{4} + \dfrac{x}{2}$; $\tan\left(\dfrac{\pi}{2} - \dfrac{x}{2}\right) = \dfrac{a}{2r} = \cot\dfrac{x}{2}$

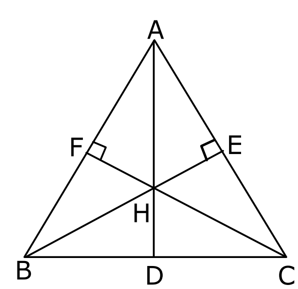


and $\tan\left(\dfrac{\pi}{4} + \dfrac{x}{4}\right) = \dfrac{a}{r} = \dfrac{1+\tan\dfrac{x}{4}}{1-\tan\dfrac{x}{4}}$

$\Rightarrow$ $\dfrac{\cos\dfrac{x}{4} + \sin\dfrac{x}{4}}{\cos\dfrac{x}{4} - \sin\dfrac{x}{4}} = \dfrac{2\cos\dfrac{x}{2}}{\sin\dfrac{x}{2}}$ $\Rightarrow$ $\sin\dfrac{x}{2} = \dfrac{2}{3}$ $\Rightarrow$ $\cos x = 1 - 2\sin^2\dfrac{x}{2} = \dfrac{1}{9}$

Ex.25 Consider an acute angled triangle ABC. Let AD, BE and CF be the altitudes drawn from the vertices to the opposite sides. Prove that, $\dfrac{EF}{a} + \dfrac{FD}{b} + \dfrac{DE}{c} = \dfrac{R+r}{R}$.

Sol. $BD = AB\cos B = c\cos B$ also $\angle BHD = \dfrac{\pi}{2} - \angle EBC = \dfrac{\pi}{2} - \left(\dfrac{\pi}{2} - C\right) = C$



$\Rightarrow$ $BH = \dfrac{BD}{\sin C} = \dfrac{c\cos B}{\sin C} = 2R\cos B$

Now the points H, D, B and F are concyclic and BH is the diameter of the circle passing through these four points. In fact this circle is also the circumcircle of triangle BFD $\Rightarrow$ $\frac{FD}{\sin B}$ = BH = 2R cos B

$\Rightarrow$ FD = 2R sin B cos B = b cos B $\Rightarrow$ $\frac{FD}{b}$ = cos B. Similarly $\frac{EF}{a}$ = cos A and $\frac{DE}{c}$ = cos C

Thus $\frac{EF}{a} + \frac{FD}{b} + \frac{DE}{c} = \cos A + \cos B + \cos C = 1 + 4 \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 1 + \frac{r}{R} = \frac{R+r}{R}$.

Ex.26 If the sum of the pairs of radii of the escribed circle of a triangle taken in order round the triangle be denoted by, s_1, s_2, s_3 and the corresponding differences by d_1, d_2, d_3.
prove that, $d_1d_2d_3 + d_1s_2s_3 + d_2s_3s_1 + d_3s_1s_2 = 0$.

Sol. $\left.\begin{array}{l} s_1 = r_1 + r_2 \\ s_2 = r_2 + r_3 \\ s_3 = r_3 + r_1 \end{array}\right]$ and $\begin{array}{l} d_1 = r_1 - r_2 \\ d_2 = r_2 - r_3 \\ d_3 = r_3 - r_1 \end{array}$. Now $s_1 = \frac{\Delta}{s-a} + \frac{\Delta}{s-b} = \Delta\left[\frac{2s-(a+b)}{(s-a)(s-b)}\right] = \frac{\Delta c}{(s-a)(s-b)}$

$d_1 = \frac{\Delta(a-b)}{(s-a)(s-b)}$ $\quad\therefore\quad$ $d_1d_2d_3 = \frac{\Delta^3(a-b)(b-c)(c-a)}{[(s-a)(s-b)(s-c)]^2}$

$d_1s_2s_3 = \frac{\Delta(a-b)}{(s-a)(s-b)}\ \frac{\Delta^2 ab}{(s-b)(s-c)(s-c)(s-a)} = \frac{\Delta^3 ab(a-b)}{[(s-a)(s-b)(s-c)]^2}$

$d_2s_3s_1 = \frac{\Delta^3(b-c)bc}{[(s-a)(s-b)(s-c)]^2}$, $d_3s_1s_2 = \frac{\Delta^3(c-a)ac}{[(s-a)(s-b)(s-c)]^2}$. Now on adding the sum vanishes

Ex.27 If the excircle touching the side c of the triangle ABC passes through its circumcentre, then prove that, $\sin A + \sin B + \sin C = (\sqrt{3}+1)\cot\frac{C}{2}$.

Sol. Distance between the circumcentre and the excentre I_3 is $d = \sqrt{R^2 + 2Rr_3}$. As the excircle passes through its circumcentre $\Rightarrow$ $d = r_3$ $\Rightarrow$ $r_3^2 = R^2 + 2Rr_3$ $\Rightarrow$ $2r_3^2 = (R + r_3)^2$

Ex.28 The radii r_1, r_2, r_3 of escribed circles of a triangle ABC are in harmonic progression. If its area is 24 sq. cm and its perimeter is 24 cm, find the lengths of its sides.

Sol. $\frac{s-a}{\Delta}, \frac{s-b}{\Delta}, \frac{s-c}{\Delta}$ are in A.P. $\Rightarrow$ a, b, c are in A.P. $\Rightarrow$ 2b = a + c $\Rightarrow$ 2s = 24 $\Rightarrow$ s = 12

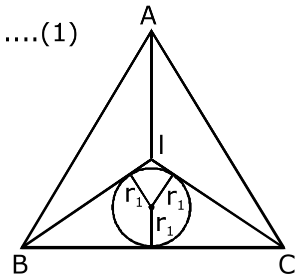
$\Rightarrow$ $\sqrt{12(12-a)4(12-16+a)} = 24$ $\Rightarrow$ $12 \times 4(12 - a)(a - 4) = 24 \times 24$ $\Rightarrow$ $-a^2 + 16a - 48 = 12$

$\Rightarrow$ $a^2 - 16a + 60 = 0$ $\Rightarrow$ $(a - 10)(a - 6) = 0$ $\Rightarrow$ a = 10, a = 6 6, 8, 10 cms

Ex.29 ABC is a triangle with I as its incentre. The radii of the incircles of the triangles BIC, AIB andAIC are r_1, r_2, and r_3 respectively. Prove that AI + BI + CI = $\frac{a(r-r_1)}{2r_1} + \frac{b(r-r_2)}{2r_2} + \frac{c(r-r_3)}{2r_3}$.

Sol. The area of the triangle BIC = Δ_1

$\Delta_1 = \frac{1}{2}ar_1 + \frac{BIr_1}{2} + \frac{CIr_1}{2} \Rightarrow \frac{1}{2}ar = \frac{1}{2}(a + BI + CI)\,r_1 \Rightarrow \frac{a(r-r_1)}{r_1} = BI + CI$ ....(1)

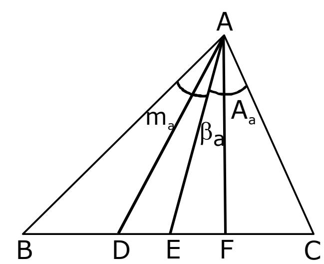


Similarly $\frac{b(r-r_2)}{r_2}$ = AI + BI and $\frac{c(r-r_3)}{r_3}$ = AI + CI ....(2)

From (1) and (2), we get AI + BI + CI = $\frac{a(r-r_1)}{2r_2} + \frac{b(r-r_2)}{2r_2} + \frac{c(r-r_3)}{2r_3}$.

J. LENGTH OF ANGLE BISECTORS, MEDIANS & ALTITUDE

(i) Length of an angle bisector from the angle A = $\beta_a = \frac{2bc\cos\frac{A}{2}}{b+c}$.

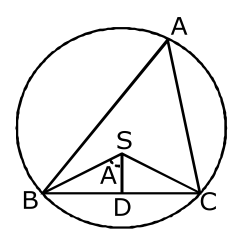


(ii) Length of median from the angle A = $m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$

(iii) Length of altitude from the angle A = $A_a = \frac{2\Delta}{a}$.

Note : $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$

To find the radius of the circle circumscribing a triangle. Let S be the centre of the circle circumscribing the triangle ABC, and R its radius. Bisect ∠BSC by SD, which will also bisect BC at right angles. ∠BSC at centre = twice ∠BAC = 2A and $\frac{a}{2}$ = BD = BS sin BSD = R sin A $\therefore$ R = $\frac{a}{2\sin A}$



Thus $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ = 2R or a = 2R sinA, b = 2R sin B, c = 2R sin C

The circum-radius may be expressed in a form not involving the angles, as

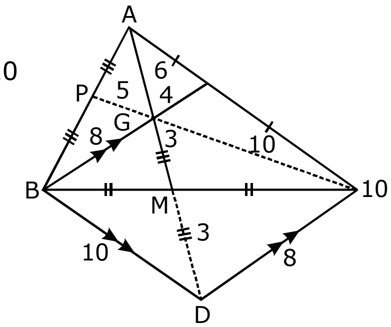
$$R = \frac{a}{2\sin A} = \frac{abc}{2bc\sin A} = \frac{abc}{4\Delta}$$

Ex.30 Show that $2R^2 \sin A \sin B \sin C = \Delta$.

Sol. The first side = $\frac{1}{2}$. 2R sin A. 2R sin B. sin C = $\frac{1}{2}$ ab sin C = Δ

Ex.31 The medians of a triangle ABC are 9 cm, 12 cm and 15 cm respectively. Then the area of the triangle is

Sol. Produce the median AM to D such that GM = MD. Join D to B and C. Now GBDC is a parallelogram. Note that the sides of the ΔGDC are 6, 8, 10 $\Rightarrow$ $\angle$GDC = 90º

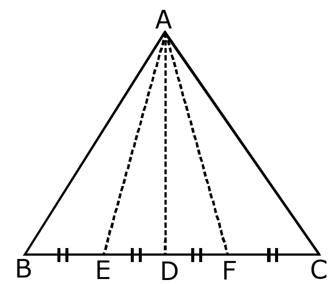


$$\left.\begin{array}{l}\text{Area of } \Delta ADC = \dfrac{12.8}{2} = 48 \\ \text{Area of } \Delta MDC = \dfrac{3.8}{2} = 12\end{array}\right]$$

$\Rightarrow$ Area of ΔAMC = 36 $\Rightarrow$ Area of ΔABC = 72 cm^2

Ex.32 AD is a median of the ΔABC. If AE and AF are medians of the triangles ABD and ADC respectively, and AD = m_1, AE = m_2, then prove that $m_2^2 + m_2^2 = \dfrac{a^2}{B}$.

Sol. In ΔABC, $AD^2 = \dfrac{1}{4}(2b^2 + 2c^2 - a^2) = m_1^2$ ....(i)



$\because$ In DAGC, $AE^2 = m_2^2 = \dfrac{1}{4}\left(2c^2 + 2AD^2 - \dfrac{a^2}{4}\right)$ ....(ii)

Similarly in ΔADC, $AF^2 = m_3^2 = \dfrac{1}{4}\left(2AD^2 + 2b^2 - \dfrac{a^2}{4}\right)$ ....(iii)

by adding equation (ii) and (iii), we get

$$\because \quad m_2^2 + m_3^2 = \frac{1}{4}\left(4AD^2 + 2b^2 + 2c^2 - \frac{a^2}{2}\right) = AD^2 + \frac{1}{4}\left(2b^2 + 2c^2 - \frac{a^2}{2}\right)$$

$$= AD^2 + \frac{1}{4}\left(2b^2 + 2c^2 - a^2 + \frac{a^2}{2}\right) = AD^2 + \frac{1}{4}(2b^2 + 2c^2 - a^2) + \frac{a^2}{8} = AD^2 + AD^2 + \frac{a^2}{8}$$

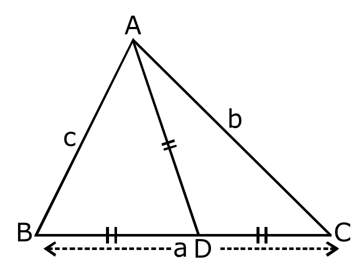
$$= 2AD^2 + \frac{a^2}{8} \quad (\because \ AD^2 = m_1^2) \qquad = 2m_1^2 + \frac{a^2}{8} \qquad \therefore \quad m_2^2 + m_3^2 - 2m_1^2 = \frac{a^2}{8}$$

Ex.33 In ΔABC, in the usual notation, the area is $\dfrac{1}{2}$ be sq. units AD is the median to BC.

Prove that $\angle ABC = \dfrac{1}{2}\angle ADC$.

Sol. $\Rightarrow$ sin A = 1 $\Rightarrow$ $\angle$A = 90º

Since AD is the median and $\angle$A = 90º, D, the midpoint of BC is the centre of the circumcircle of ΔABC.

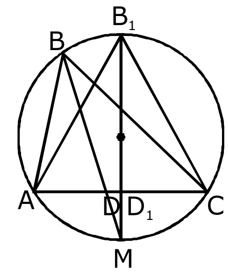


So AD = BD = DC $\Rightarrow$ $\angle ABC = \dfrac{1}{2}\angle ADC$

(angle subtended by AC at the circumference = $\dfrac{1}{2}$ angle subtended by AC at the centre).

Ex.34 Prove that of all the triangles with a given base and a given vertex angle, an isosceles triangle has the greatest bisector of the vertex angle.

Sol. Let us give a geometrical proof which is considerably briefer and more elegant than the first method.

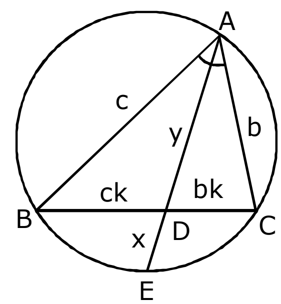


Circumscribe a circle above the triangle ABC with the angle bisector BD (fig.). The vertices of all the rest of triangles with a given base and a given vertex angle lie on the arc ABC. Let us take an isosceles triangle AB_1C, draw the angle bisector B_1D_1 in it, and prove that BD < B_1D_1 in it, and prove that BD < B_1D_1.

Extend both angle bisectors BD and B_1D_1 to intersect the circle. Both of them will intersect the circle. Both of them will intersect the circle at one and the same point M which is the midpoint of the arc AC. Since B_1M is a diameter of the circle, we have : BM < B_1M. From the triangle DD_1M. From these inequalities it follows that BM – DM < B_1M – D_1M, that is BD < B_1D_1.

Ex.35 In a ΔABC, the bisector of the angle A meets the side BC in D and the circumscribed circle in E. Show

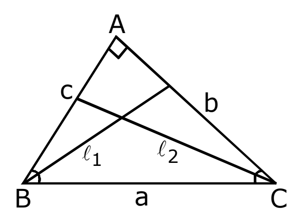
that, DE = $\dfrac{a^2 \sec\frac{A}{2}}{2(b+c)}$.



Sol. ck + bk = a $\Rightarrow$ k = $\dfrac{a}{b+c}$ also xy = bck^2

$$\Rightarrow \quad x \,.\, \frac{2b\cos\frac{A}{2}}{b+c} = \frac{bca^2}{(b+c)^2} \quad \Rightarrow \quad x = \frac{a^2\sec\frac{A}{2}}{2(b+c)} = DE$$

Ex.36 The ratios of the lengths of the sides BC & AC of a triangle ABC to the radius of a circumscribed circle are equal to 2 & 3/2. respectively. Show that the ratio of the lengths of the bisectors of the interior angles B & C is, $\dfrac{7(\sqrt{7}-1)}{9\sqrt{2}}$.



Sol. $\dfrac{a}{R} = 2$; $\dfrac{b}{R} = \dfrac{3}{2}$ $\quad\therefore\quad$ $\dfrac{2R\sin A}{R} = 2$; $\quad$ sin B = $\dfrac{3}{4}$

$$\Rightarrow \sin A = 1 \ ; \ c^2 = 4R^2 - \frac{9R^2}{4} \ \Rightarrow \ A = 90^\circ \ ; \ c = \frac{\sqrt{7}}{2}R$$

Now $I_1 = \dfrac{2ac}{a+c}\cos\dfrac{B}{2} = \dfrac{2ac}{a+c}\sqrt{\dfrac{1+\cos B}{2}}$ and $I_2 = \dfrac{2ab}{a+b}\cos\dfrac{c}{2} = \dfrac{2ab}{a+b}\sqrt{\dfrac{1+\cos C}{2}}$

$$\therefore \quad \frac{I_1}{I_2} = \frac{a+b}{a+c} \,.\, \frac{c}{b}\sqrt{\frac{1+\cos B}{1+\cos C}} = \frac{c(a+b)}{b(a+c)}\sqrt{\frac{1+\frac{c}{a}}{1+\frac{b}{a}}} = \frac{c}{b}\sqrt{\frac{a+b}{a+c}}$$

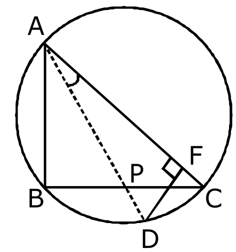
Substituting a = 2 R ; b = $\dfrac{3}{2}$ R & c = $\dfrac{\sqrt{7}}{2}$ R, we get the desired result.

Ex.37 The internal bisector of $\angle A$ of triangle ABC meets the circumcircle of the triangle in D. If DE and DF and the altitudes drawn from D to sides ABC an AC respectively, prove that AE + AF = b + c.

Sol. We have $\angle DAF = \frac{A}{2}$ and $AF = AD \cos\frac{A}{2}$ .Similarly $AE = AD\cos\frac{A}{2} \Rightarrow AF + AE = 2AD\cos\frac{A}{2}$

We also have AP . PD = BP . PC $\Rightarrow$ $AP(AD - AP) = BP \cdot PC = \frac{ac}{b+c}\cdot\frac{ba}{b+c} = \frac{a^2bc}{(b+c)^2}$

Now length of internal bisector $AP = \frac{2bc}{(b+c)}\cos\frac{A}{2}$



Thus we have, $AD - AP = \frac{a^2bc(b+c)}{(b+c)^2.2bc\cos\frac{A}{2}} = \frac{a^2}{2(b+c)\cos\frac{A}{2}}$

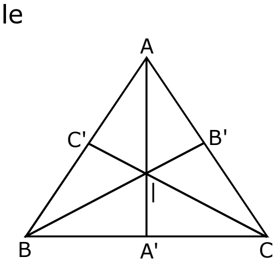
$$\Rightarrow AD = AP + \frac{a^2}{2(b+c)\cos\frac{A}{2}} = \frac{2bc\cos\frac{A}{2}}{(b+c)^2} + \frac{a^2}{2(b+c)\cos\frac{A}{2}} = \frac{4bc\cos^2\frac{A}{2}+a^2}{2(b+c)\cos\frac{A}{2}}$$

$$\Rightarrow 2AD\cos\frac{A}{2} = \frac{2bc(1+\cos A)+a^2}{b+c} = \frac{2bc\left(1+\frac{b^2+c^2-a^2}{2bc}\right)+a^2}{b+c} = (b+c)$$

Ex.38 In a ΔABC internal angle bisector AI, BI and CI are produced to meet opposite sides in A', B' and C' respectively. Prove that the maximum value of $\frac{AIBICI}{AA'.BB'.CC'}$ is $\frac{8}{27}$.

Sol. Since angle bisector divides opposite side in ratio of sides containing the angle

$\Rightarrow BA' = \frac{ac}{b+c} \Rightarrow CA' = \frac{ab}{a+c}$



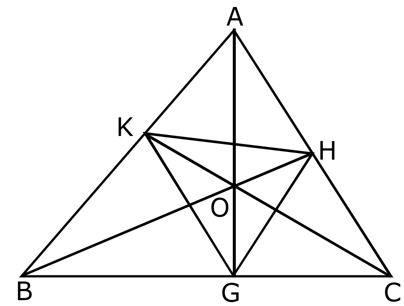
Now BI is also angle bisector of $\angle B$ for triangle ABA'

$\Rightarrow \frac{AI}{AI'} = \frac{b+c}{a} \Rightarrow \frac{AI}{AA'} = \frac{b+c}{a+b+c}$

Similarly $\frac{BI}{BI'} = \frac{a+c}{a+b+c}$ and $\frac{CI}{CC'} = \frac{a+b}{a+b+c}$ $\Rightarrow$ $\frac{AI.BI.CI}{AA'.BB'.CC'} = \frac{(a+b)(b+c)(c+a)}{(a+b+c)^3}$

Using A.M. $\ge$ G.M. we get $\frac{2(a+b+c)}{3(a+b+c)} \ge \left(\frac{(a+b)(b+c)(c+a)}{(a+b+c)^3}\right)^{1/3}$ $\Rightarrow$ $\frac{(a+b)(b+c)(c+a)}{(a+b+c)^2} \le \frac{8}{27}$.

K. ORTHOCENTRE AND PEDAL TRIANGLE



Let G, H, K be the feet of the perpendiculars from the angular points on the opposite sides of the triangle ABC, then GHK is called the Pedal triangle of ABC. The three perpendiculars AG, BH, CK meet in a point O which is called the orthocentre of the triangle ABC.

To find the sides and angles of the pedal triangle.

In the figure, the points K, O, G, B are concyclic : $\therefore$ $\angle OGK = OBK = 90^o - A$

Also the points H, O, G, C are concyclic : $\therefore$ $\angle OGH = \angle OCH = 90^o - A$

$\therefore$ $\angle KGH = 180^o - 2A$

Thus the angles of the pedal triangle are $180^o - 2A$, $180^o - 2B$, $180^o - 2C$

Again, the triangles AKH, ABC are similar : $\therefore$ $\dfrac{HK}{BC} = \dfrac{AK}{AC} = \cos A$ $\therefore$ $HK = a \cos A$

Thus the sides of the pedal triangle are a cos A, b cos B, c cos C.

In terms of R, the equivalent forms become R sin 2A, R sin 2B, R sin 2C.

If the angle ACB of the given triangle is obtuse, the expression $180^o - 2C$, and c cos C are both negative, and the values we have obtained required some modification. We have the student to show that in this case the angles are 2A, 2B, $2C - 180^o$, and the sides a cos A, b cos B, – c cos C.

Remarks :

(i) The distances of the orthocentre from the angular points of the ΔABC are 2 R cos A, 2R cos B and 2R cos C.

(ii) The distances of P from sides are 2R cos B cos C, 2R cos C cos A & 2R cos B cos C.

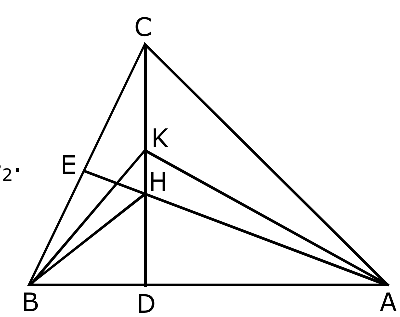
(iii) Circumradii of the triangles PBC, PCA, PAB and AABC are equal.

To find the area and circum-radius of the pedal triangle.

$$\text{Area} = \frac{1}{2}\,(\text{product of two sides}) \times (\text{sine of included angle}) = \frac{1}{2} R \sin 2B \,.\, R \sin 2C \,.\, \sin (180^o - 2A)$$

$$= \frac{1}{2} R^2 \sin 2A \sin 2B \sin 2C. \text{ The circum-radius } = \frac{HK}{2\sin HGK} = \frac{2\sin 2A}{2\sin(180^o - 2A)} = \frac{R}{2}.$$

Ex.39 The point H is the orthocentre of the triangle ABC. A point K is taken on the straight line CH such that ABK is a right triangle. Prove that the area of the triangle ABK is the geometric mean between the area of the triangles ABC and ABH.



Sol. We introduce the following notation : $S_{ABK} = S$, $S_{ABC} = S_1$, $S_{ABH} = S_2$.

Then $S = \dfrac{1}{2}\,AB \,.\, KD$, $S_1 = \dfrac{1}{2}\,AB \,.\, CD$, and $S_2 = \dfrac{1}{2}\,AB \,.\, HD$.

We have to prove that $S = \sqrt{S_1 S_2}$, ...(1)

i.e. $\dfrac{1}{2}\,AB \,.\, KD = \sqrt{\dfrac{1}{2} AB.CD.\dfrac{1}{2} ABHD}$ or $KD^2 = CD \,.\, HD.$ ....(2)

But ABK is a triangle, and therefore, KD^2 = BD. AD. Thus, equality (2) will be ascertained if we prove that BD . AD = CD . DH, or that $\frac{BD}{CD} = \frac{DH}{AD}$. The last equality obviously follows from and HDA (in these triangles the angles BCD and HAD are equal as angles with mutually perpendicular sides since AE is the altitude of the triangle). Hence, Equality (2) as well as equality (1) have been proved.

Ex.40 If f, g, h denote sides, the pedal triangle of a ΔABC, then show that $\frac{(b^2 - c^2)}{a^2} + \frac{(c^2 - a^2)}{b^2} + \frac{(a^2 - g^2)h}{c^2} = 0$

Sol. Sides are a cos A, b cos B, c cos C. Hence LHS $\left(\frac{b^2 - c^2}{a}\right)$ cos A + $\left(\frac{c^2 - a^2}{b}\right)$ cos B + $\left(\frac{a^2 - b^2}{c}\right)$ cos C

Put the values of cos A etc get the result.

Ex.41 Vertex A of a variable triangle ABC, inscribed in a circle of radius R, is a fixed point. If the angles subtended by the side BC at orthocentre (H), circumcentre (O) and incentre (I) are equal than identify the locus of orthocentre of triangle ABC.

Sol. The angles subtended by the side BC at points H, O and I are B + C, 2A and $\pi - \left(\frac{B+C}{2}\right)$ respectively.

$\Rightarrow$ B + C = 2A = 180 – $\left(\frac{B+C}{2}\right)$ $\Rightarrow$ A = $\frac{\pi}{3}$ and B + C = $\frac{2\pi}{3}$. Also in triangle ABC, HA = 2R cos A = R

$\Rightarrow$ HA is contant. $\Rightarrow$ locus of orthocentre is a circle having centre at the vertex A.

L. EXCENTRAL TRIANGLE

Let ABC be a triangle I_1, I_2, I_3 its ex-centres ; then $I_1I_2I_3$ is called the Ex-central triangle of ABC. Let I be the in-centre ; then from the construction for finding the positions of the in-centre and ex-centres, it follows that :

(i) The points I, I_1 lie on the line bisecting the angle BAC; the points I, I_2 lie on the line bisecting the angle ABC; the points I, I_3 lie on the line bisecting the angle ACB.

(ii) The points I_2, I_3 lie on the line bisecting the angle BAC externally; the points I_3,I_1 lie on the line bisecting the angle ABC externally ; the points I_1, I_2 lie on the line bisecting the angle ABC externally.

(iii) The line AI_1 is perpendicular to I_2I_3; the line BI_2 is perpendicular to I_3I_1 ; the line CI_3 is perpendicular to I_1I_2. Thus the triangle ABC is the Pedal triangle of its ex-central triangle $I_1I_2I_3$.

(iv) The angles IBI_1 and ICI_1 are right angles; hence the points B, I, C, I_1 are concyclic . Similarly, the points C, I, A, I_2, and the points A, I, B, I_3 are concyclic.

(v) The lines AI_1, BI_2, CI_3 meet at the in-centre I, which is therefore the orthocentre of the ex-central triangle $I_1I_2I_3$.

(vi) The lines AI_1, BI_2, CI_3 meet at the in-centre I, which is therefore the orthocentre of the ex-central triangle $I_1I_2I_3$.

(vii) Each of the four points I, I_1,I_2,I_3 is the orthocentre of the triangle formed by joining the other three points.

To find the sides and angles of the ex-central triangle. With the figure of the-last article.

$\angle BI_1C = \angle BI_1I + \angle CI_1I = \angle BCI + \angle CBI = \frac{C}{2} + \frac{B}{2} = 90^\circ - \frac{A}{2}$. Thus the angles are $90^\circ - \frac{A}{2}$ $90^\circ - \frac{B}{2}$, $90^\circ - \frac{C}{2}$.

Again, the points B, I_3, I_2, C are concyclic.

$\therefore \quad \angle I_1I_2I_3 = I_3BC = \angle I_1BC \qquad \therefore$ the triangles $I_1I_2I_3$ = supplement of $\angle I_1BC$ are similar

$$\therefore \quad \frac{I_2I_3}{BC} = \frac{I_3I_1}{I_1C} = \sec\left(90^\circ \frac{A}{2}\right) = \text{cosec } \frac{A}{2} \qquad \therefore \quad I_2I_3 = a \text{ cosec } \frac{A}{2} = 4R \cos \frac{A}{2}$$

Thus the sides are $4R \cos \frac{A}{2}$, $4R \cos \frac{B}{2}$, $4R \cos \frac{C}{2}$

To find the area and circum-radius of the ex-central triangle.

The area $= \frac{1}{2}$ (product of two sides) × (sine of included angle)

$$= \frac{1}{2} \times 4R \cos \frac{B}{2} \times 4R \cos \frac{C}{2} \times \sin\left(90^\circ - \frac{A}{2}\right) = 8R^2 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

$$\text{The circum-radius} = \frac{I_2I_3}{2 \sin I_2I_1I_3} = \frac{4R \cos \frac{A}{2}}{2 \sin\left(90^\circ - \frac{A}{2}\right)} = 2R$$

To find the distances between the in-centre and ex-centres.

The $\angle IBI_1$, $\angle ICI_1$ are right angles $\quad \therefore \quad II_1$ is the diameter of the circum-circle of the triangle BCI_1

$\therefore \quad II_1 = \frac{BC}{\sin BI_1C} = \frac{a}{\cos \frac{A}{2}} = 4R \sin \frac{A}{2}$. Thus the distances are $4R \sin \frac{A}{2}$, $4R \sin \frac{B}{2}$, $4R \sin \frac{C}{2}$

We have proved that OG, OH, OK bisect the angles HGK, KHG, GHK respectively, so that O is the in-centre of the triangle GHK. Thus the orthocentre of a triangle is the in-centre of the pedal triangle. Again, the line CGB which is at right angles to OG bisect $\angle$HGK externally. Similarly the lines AHC and BKA bisect $\angle$KHG and $\angle$GKH externally, so that ABC is the ex-central triangle of its pedal triangle GHK.

Ex.42 Show that the sides, angles and area of an excentral triangle of ΔABC are,

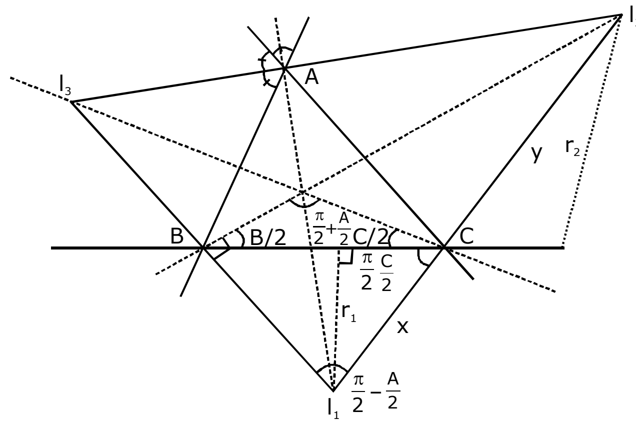
$$\left(4R \cos \frac{A}{2}, 4R \cos \frac{B}{2}, 4R \cos \frac{C}{2}\right); \left(\frac{\pi}{2} - \frac{A}{2}, \frac{\pi}{2} - \frac{B}{2}, \frac{\pi}{2} - \frac{C}{2}\right)$$

and 2RS respectively where all symbols have their usual meaning.

Sol. Angle of D I_1, I_2, I_3 are obviously $\frac{\pi}{2} - \frac{A}{2}$, $\frac{\pi}{2} - \frac{B}{2}$ & $\frac{\pi}{2} - \frac{C}{2}$ respectively.

note that IBI_1C is a cyclic quadrilateral. $II_1 = \frac{BC}{\cos\frac{A}{2}} = \frac{a}{\cos\frac{A}{2}} = \frac{2R\sin A}{\cos\frac{A}{2}} = 4R\cos A/2$

Now $I_1I_2 = x + y = \frac{r_1 + r_2}{\cos\frac{C}{2}} = \frac{s\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right)}{\cos\frac{C}{2}}$



$x + y = \frac{s\sin\left(\frac{A}{2}+\frac{B}{2}\right)}{II\cos\frac{A}{2}} = \frac{s\cos\frac{C}{2}}{II\cos\frac{A}{2}}$

$= \frac{4s\cos\frac{C}{2}}{\Sigma\sin A} = \frac{4s\cos\frac{C}{2}2R}{a+b+c} = 4R\cos\frac{C}{2}$

$\Rightarrow$ sides opposite I_1, I_2, I_3 are $4R\cos\frac{A}{2}$, $4R\cos\frac{B}{2}$, $4R\cos\frac{C}{2}$ respectively

Area of $I_1I_2I_3 = \frac{1}{2}\ 4R\cos\frac{A}{2}, 4R\cos\frac{B}{2}\sin\left(\frac{\pi}{2}-\frac{C}{2}\right) = 8R^2\pi\cos\frac{A}{2} = 2R^2(\Sigma\sin A) = 2R^2\left[\frac{a+b+c}{2R}\right] = 2RS$

Ex.43 If I is the incentre and I_1, I_2, I_3 are the centre of escribed circles of the ΔABC, Prove that

(i) $II_1, II_2, II_3 = 16R^2r$ **(ii)** $II_1^2 + I_2I_3^2 = II_2^2 + I_3I_1^2 = I_1I_2^2 + II_3^2$

Sol. **(i)** $\because$ We know that $II_1 = a\sec\frac{A}{2}$, $II_2 = b\sec\frac{B}{2}$ and $II_3 = c\sec\frac{C}{2}$

$\because$ $I_1I_2 = c, \operatorname{cosec}\frac{C}{2}$, $I_2I_3 = a\operatorname{cosec}\frac{A}{2}$ and $I_3I_1 = b\operatorname{cosec}\frac{B}{2}$

$\because$ $II_1.II_2.II_3 = abc\sec\frac{A}{2}\sec\frac{B}{2}\sec\frac{C}{2}$ ....(i)

$\because$ $a = 2R\sin A$, $b = 2R\sin B$ and $c = 2R\sin C$

$\therefore$ equation (i) becomes $\because$ $II_1.II_2.II_3 = (2R\sin A)(2R\sin B)(2R\sin C)\sec\frac{A}{2}\sec\frac{B}{2}\sec\frac{C}{2}$

$= 8R^3 . \frac{\left(2\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(2\sin\frac{B}{2}\cos\frac{B}{2}\right)\left(2\sin\frac{C}{2}\cos\frac{C}{2}\right)}{\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}} = 64R^2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$\because$ $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ $\therefore$ $II_1.II_2.II_3 = 16R^2r$

(ii) $II_1^2 + I_2I_3^2 = II_2^2\ I_3I_1^2 = II_3^2 + I_1I_2^2$

$\because \quad II_1^2 + I_2I_3^2 = a^2 \sec^2\frac{A}{2} + a^2 . \operatorname{cosec}^2\frac{A}{2} = \dfrac{a^2}{\sin^2\frac{A}{2}\cos^2\frac{A}{2}}$

$\because \quad a = 2\ R \sin A = 4R \sin\frac{A}{2}\cos\frac{A}{2} \quad \therefore \quad II_1^2 + I_2I_3^2 = \dfrac{16R^2\sin^2\frac{A}{2}.\cos^2\frac{A}{2}}{\sin^2\frac{A}{2}.\cos^2\frac{A}{2}} = 16R^2$

Hence $II_1^2 + I_2I_3^2 = II_2^2 + I_3I_1^2 = II_3^2 + I_1I_2^2$

M. DISTANCES OF SPECIAL POINTS FROM VERTICES AND SIDES OF A TRIANGLE

(i) Circumcentre (O) : $OA = R\ \&\ O_a = R\cos A$

(ii) Incentre (I) : $IA = r\operatorname{cosec}\frac{A}{2}\ \&\ I_a = r$

(iii) Excentre (I_1) : $I_1A = r_1\operatorname{cosec}\frac{A}{2}\ \&\ I_{1a} = r_1$

(iv) Orthocentre (H) : $HA = 2R\cos A\ \&\ H_a = 2R\cos B\cos C$

(v) Centrod (G) : $GA = \frac{1}{3}\sqrt{2b^2 + 2c^2 - a^2}\ \&\ G_a = \frac{2\Delta}{3a}$

N. DISTANCES BETWEEN SPECIAL POINTS

(a) The distance between circumcentre and orthocentre is $= R . \sqrt{1 - 8\cos A\cos B\cos C}$

(b) The distance between circumcentre and incentre is $= \sqrt{R^2 - 2Rr} = R\sqrt{1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$

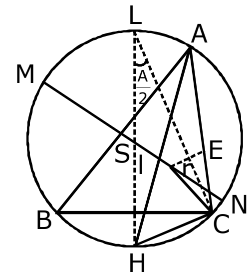
(c) Distance between circumcentre and centroid $OG = \sqrt{R^2 - \frac{1}{9}(a^2 + b^2 + c^2)}$

(d) The distance between incentre and orthocentre is $\sqrt{2r^2 - 4R^2\cos A\cos B\cos C}$

To find the distance between the in-centre and circum-centre.

Let S be the circum-centre and I the in-centre.Produce Al to meet the circum-circle in H; join CH and Cl. Draw IE perpendicular to AC. Produce HS to meet the circumference in L, and join CL. Then

$\angle HIC = \angle IAC + \angle ICA = \frac{A}{2} + \frac{C}{2}$;



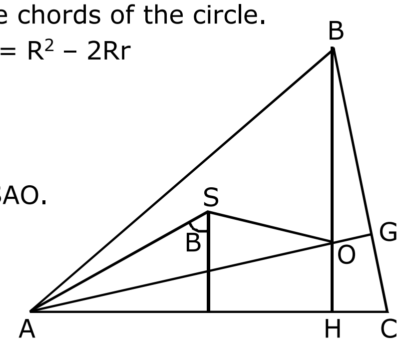
$\angle HCI = \angle ICB + \angle BCH = \frac{C}{2} + \angle BAH = \frac{C}{2} + \frac{A}{2}$;

$\therefore$ $\angle HCI = HIC$; $\therefore$ $HI = HC = 2R \sin \frac{A}{2}$.

Also $AI = IE \text{ cosec } \frac{A}{2} = r \text{ cosec } \frac{A}{2}$ $\therefore$ $AI \,.\, IH = 2Rr$

Produce SI to meet the circumference in M and N. Since MIN, AIH are chords of the circle.

$AI.IH = MI.\,IN = (R + SI)\,(R - SI)$; $\therefore$ $2Rr = R^2 - SI^2$; $SI^2 = R^2 - 2Rr$



To find the distance of the orthocentre from the circum-centre.

With the usual notation, we have $SO^2 = SA^2 + AO^2 - 2SA\,.\,AO \cos SAO$.

Now $AS = R$; $AO = AH \text{ cosec } C = c \cos A \text{ cosec } C$

$= 2R \sin C \cos A \text{ cosec } C = 2R \cos A$;

$\angle SAO = \angle SAC - \angle OAC = (90º - B) - (90º - C) = C - B$

$\therefore$ $SO^2 = R^2 + 4R^2 \cos^2 A - 4R^2 \cos A \cos (C - B)$

$= R^2 - 4R^2 \cos^2 A \{\cos (B + C) + \cos (C - B)\} = R^2 - 8R^2 \cos A \cos B \cos C$

Ex.44 If r and R are radii of the incircle and circumcircle of ΔABC, prove that

$$8rR \left(\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2}\right) = 2\,bc + 2\,ca + 2ab - a^2 - b^2 - c^2$$

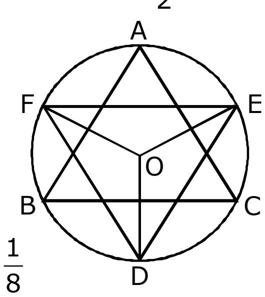
Sol. $LHS = \frac{8\Delta s}{s} \, \frac{abc}{4\Delta} \, \Sigma \cos^2 \frac{A}{2} = \frac{abc}{s}\left[\Sigma 2 \cos^2 \frac{A}{2}\right] = \frac{abc}{s}\,[(1 + \cos A) + (1 + \cos B) + (1 + \cos C)]$

$= \frac{abc}{s}\left[\frac{(b+c)^2 - a^2}{2bc} + \frac{(c+a)^2 - b^2}{2ca} + \frac{(a+b)^2 - c^2}{2ab}\right] = \frac{abc}{s}\,\frac{(2s)}{2abc}\,[a(b + c - a) + b\,(c + a - b) + c(a + b - c)]$

Ex.45 The perpendicular bisectors of sides of triangle ABC intersect its circumcircle at D, E and F respectively. If the area of triangle ABC is equal to the area of triangle DEF then prove that the triangle ABC and DEF are equilateral.

Sol. The triangle ABC and DEF have the same circumcircle.Let the radius of the circumcircle be R

$\Rightarrow$ Area of $\Delta ABC = 2R^2 \sin A \sin B \sin C$ ....(1) Also $\angle DOF = 180º - B \Rightarrow \angle DEF = 90º - \frac{B}{2}$



Similarly $\angle DFE = 90º - \frac{C}{2}$ and $\angle FDE = 90º - \frac{A}{2}$

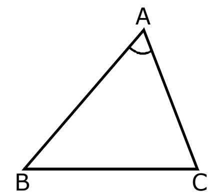
$\Rightarrow$ Area of $\Delta DEF = 2R^2 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$ ....(2)

from (1) and (2), $\sin A \sin B \sin C = \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2} \Rightarrow \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \frac{1}{8}$

$\Rightarrow \Delta ABC$ is equilateral also $\angle DEF = 90º - \frac{B}{2} = 60º$. Similarly $\angle DEF = \angle FED = 60º \Rightarrow$ ΔDEF is also equilateral.

Ex.46 If the angle A of triangle ABC is $\frac{\pi}{3}$, then prove that the vertices B, C orthocentre, circumcentre and incentre are concylic.

Sol. The angle subtended by the side BC at the orthocentre, the circumcentre and the incentre are B + C, 2A and 90 + $\frac{A}{2}$ respectively. If $\angle A = 60^o$, then $B + C = 2A = 90 + \frac{A}{2} = 120^o$



$\Rightarrow$ Angle subtended by BC at orthocentre, circumcentre and incentre are equal

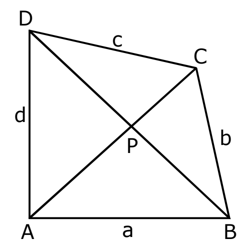
$\Rightarrow$ B, C, orthocentre, circumcentre and incentre are angle C.

O. QUADRILATERAL

To prove that the area of a quadrilateral is equal to $\frac{1}{2}$ (product of the diagonals) × (sine of included angle).

Let the diagonals AC, BD intersect at P, and let $\angle DPA = \alpha$, and let S denote the area of the quadrilateral.

$\Delta DAC = \Delta APD + \Delta CPD$



$= \frac{1}{2}$ DP . AP sin α + $\frac{1}{2}$DP . PC sin $(\pi - \alpha)$

$= \frac{1}{2}$ DP (AP + PC) sin a = $\frac{1}{2}$ DP.AC sin α

Similarly $\Delta ABC = \frac{1}{2}$BP.AC sin α $\therefore$ S = $\frac{1}{2}$ (DP + BP) AC sin α = $\frac{1}{2}$ DB . AC sinα

Ex.47 A quadrilateral ABCD is such that one circle can be inscribed in it and another circle circumscribed about it ; show that $\tan^2 \frac{A}{2} = \frac{bc}{ad}$.

Sol. If a circle can be inscribed in a quadrilateral, the sum of one pair of the opposite sides is equal to that of the other pair : a + c = b + d. Since the quadrilateral is cyclic, $\cos A = \frac{a^2 + d^2 - b^2 - c^2}{2(ad + bc)}$.

But a – d = b – c, so that $a^2 - 2ad + d^2 = b^2 - 2bc + c^2$;

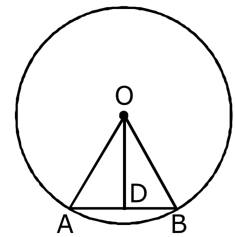
$\therefore$ $a^2 + d^2 - b^2 - c^2 = 2(ad - bc)$ $\therefore$ $\cos A = \frac{ad - bc}{ad + bc}$ $\therefore$ $\tan^2\frac{A}{2} = \frac{1 - \cos A}{1 + \cos B} = \frac{bc}{ad}$

P. INSCRIBED AND CIRCUMSCRIBED POLYGONS

To find the perimeter and area of a regular polygon of n sides inscribed a circle.

Let r be the radius of the circle, and AB a side of the polygon.
Join OA, OB, and draw OD bisecting $\angle$AOB; then AB is bisected at right angles in D.

And $\angle AOB = \frac{1}{n}$ (four right angles) $= \frac{2\pi}{n}$



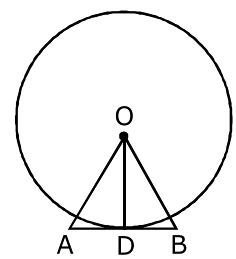
Perimeter of polygon = nAB = 2nAD = 2nCA sin AOD = $2nr \sin \frac{\pi}{n}$

Area of polygon = n (area of triangle AOB) = $\frac{1}{2} nr^2 \sin \frac{2\pi}{n}$.

To find the perimeter and area of a regular polygon of n sides circumscribed about a given circle.

Let r be the radius of the circle, and AB a side of the polygon. Let AB touch the circle at D. Join OA, OB, OD; then OD bisects AB at right angles, and also bisects $\angle$AOB.

=nAB = 2nAD = 2nOD tan AOD = $2nr \tan \frac{\pi}{n}$.



Area of polygon = n (area of triangle AOB)

= nOD . AD = $nr^2 \tan \frac{\pi}{n}$

Q. MISCELLANEOUS QUESTIONS

Ex.48 If a, b, c denote the sides of a ΔABC, show that the value of the expression,
$a^3 (p - q)(p - r) + b^2 (q - r) + b^2 (q - r)(q - p) + c^2 (r - p)(r - q)$ cannot be negative where p, q, r $\in$ R.

Sol. Let p > q > r and $\left.\begin{matrix} p - q = y > 0 \\ q - r = z > 0 \end{matrix}\right\} \Rightarrow p - r = y + z > 0$

Consider $E = a^2y(y + z) - b^2zy + c^2z(y + z) = a^2y^2 + c^2z^2 + yz(a^2 + c^2 - b^2)$
Now $b < a + c \Rightarrow b_{max} = a + c$
$E_{least} = a^2y^2 + c^2z^2 + yz[a^2 + c^2 - (a + c)^2] = a^2y^2 + c^2z^2 - 2acyz = (ay - cz)^2 \geq 0$

Ex.49 In a triangle ABC is $2 \cos \frac{B}{2} \cos \frac{C}{2} = \frac{1}{2} + \left(\frac{b+c}{a}\right) \sin \frac{A}{2}$ then find the measure of angle A.

Sol. Given $2 \cos \frac{B}{2} \cos \frac{C}{2} = \frac{1}{2} + \left(\frac{b+c}{a}\right) \sin \frac{A}{2}$ or $\cos\left(\frac{B+C}{2}\right) + \cos\left(\frac{B-C}{2}\right) = \frac{1}{2} + \left(\frac{\sin B + \sin C}{\sin A}\right) \sin \frac{A}{2}$

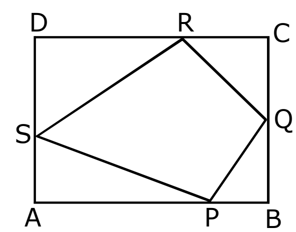
or $\sin \frac{A}{2} + \cos\left(\frac{B-C}{2}\right) = \frac{1}{2} + \frac{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)\sin\frac{A}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}}$

or $\sin\frac{A}{2} + \cos\left(\frac{B-C}{2}\right) = \frac{1}{2} + \frac{\cos\frac{A}{2}\cos\left(\frac{B-C}{2}\right)}{\cos\frac{A}{2}}$ or $\sin\frac{A}{2} + \cos\left(\frac{B-C}{2}\right) = \frac{1}{2} + \cos\left(\frac{B-C}{2}\right)$

$\Rightarrow \quad \sin\frac{A}{2} = \frac{1}{2} \qquad \therefore \quad \angle A/2 = 30^o \Rightarrow \angle A = 60^o$

Ex.50 Suppose ABCD is a rectangle and P,Q,R,S are points on the sides AB, BC, CD, DA respectively.
Show that PQ + QR + RS + SP $\geq \sqrt{2}$ AC.

Sol. We have (see figure) PQ. QR > BQ . QC, QR . RS > CR . RD, etc.

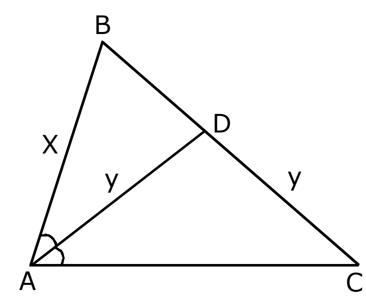


Therefore, $(PQ + QR + RS + SP)^2 = PQ^2 + ... + 2PQ . QR + ...$
$> (PB^2 + BQ^2) + 2BQ . QC + ...$
$= (PA + PB)^2 + (BQ + QC)^2 + (CR + RD)^2 + (DS + SA)^2$
$= AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 = 2AC^2$
Hence PQ + QR + RS + SP > $\sqrt{2}$ AC.

Ex.51 In a triangle ABC the angle A is twice the angle C, the side BC is 2 cm longer than the side AB, and AC = 5 cm. Find AB and BC.

Sol. Drawing the bisector AD of the angle A, we get : $\angle BAD = \angle DAC = \angle ACB$,
In a triangles ADC, the base angles are equal to each other, and hence, this is an isosceles triangles : AD = DC. Setting AB = x and AD = DC = y, we find the BC = x + 2 and BD = x + 2 – y.
The triangles ABD and ABC are similar, since $\angle BAD = \angle BCA$ and $\angle B$ is a common angle. From the similarity of these triangles we conclude that $\frac{AB}{BC} = \frac{BD}{AB} = \frac{AD}{AC}$, i.e., $\frac{x}{x+2} = \frac{x+2-y}{x} = \frac{y}{5}$.

For finding x & y we have system of two equation



in two variables : $\begin{cases} \frac{x}{x+2} = \frac{y}{5}, \\ \frac{x+2-y}{x} = \frac{y}{5} \end{cases}$, hence $\begin{cases} 5x = xy + 2y, \\ 5x + 10 - 5y = xy \end{cases}$

Subtracting the second equation from the first, we get : 5y – 10 = 2y

and $y = \frac{10}{3}$. Hence, $\frac{x}{x+2} = \frac{2}{3}$, i.e. x = 4

Ex.52 In triangle ABC, cos A . cos B + cos B. cos C + cos C, cos A = 1 – 2cos A . cos B . cos C. Prove that it is possible if and only if ΔABC is equilateral.

Sol. Σ cosA . cos B = 1 – 2 cos A. cos B. cos C = 1 – cos C (cos (A + B) + cos (A – B))
$= 1 + \cos^2 A - \sin^2 B + \cos^2 C = \cos^2 A + \cos^2 B + \cos^2 C = \Sigma \cos^2 A$
$= 1 + \cos^2 A - \sin^2 B + \cos^2 C = \cos^2 + \cos^2 B + \cos^2 B + \cos^2 C = \Sigma \cos^2 A$

Thus we have, $2\Sigma \cos^2 A - 2S \cos A . \cos B = 0$

$\Rightarrow$ $(\cos A - \cos B)^2 + (\cos B - \cos C)^2 + (\cos C - \cos A)^2 = 0$

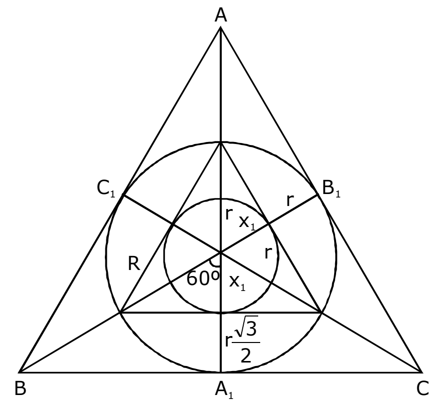
$\Rightarrow$ $\cos A = \cos B = \cos C$ $\Rightarrow$ $\angle A = \angle B = \angle C$. Thus triangle ABCis equilateral.

Now if Δ is equilateral $\angle A = \angle B = \angle C = \frac{\pi}{3} \Rightarrow \Sigma \cos A \cos B = \frac{3}{4}$ and $1 - 2 \cos A \cos B \cos C = 1 - \frac{2}{8} = \frac{3}{4}$.

Hence the given expression is true if and only if ΔABC is equilateral.

Ex.53 A circle is inscribed in an equilateral triangle ABC; an equilateral triangle in the circle, a circle again in the latter triangle and so on; in this way (n + 1) circles are described ; if r, x_1, x_2, ..., x_n be the radii of the circles, show that $r = x_1 + x_2 + x_3 + ... + x_{n-1} + 2x_n$

Sol. For equilateral $\Delta = r = \frac{R}{2}$. Now r is the circum radius for $\Delta A_1B_1C_1$.



Hence $x_1 = \frac{r}{2} = \frac{R}{2^2}$. Similarly $x_2 = \frac{x_1}{2} = \frac{R}{2^3}$

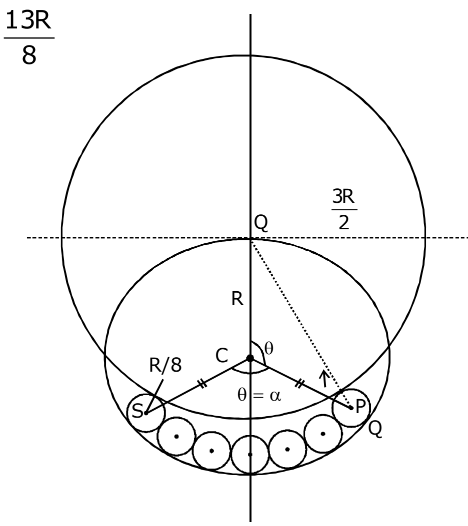
Now $x_1 + x_2 + ... + x_{n-1} + x_n$

$$= \frac{R}{2^2}\left[1 + \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^{n-1}}\right] + \frac{R}{2^{n+1}}$$

$$= \frac{R}{2^2}\left[\frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}}\right] = \frac{R}{2}\left[1 - \frac{1}{2^n}\right] + \frac{R}{2^{n+1}} = \frac{R}{2} - \frac{R}{2^{n+1}} + \frac{R}{2^{n+1}} = r$$

Ex.54 A point 'O' is situated on a circle of radius R and with centre O, another circle of radius $\frac{3R}{2}$ is described inside the crescent shaped area intercepted between these circles, a circle of radius R/8 is placed. If the same circle moves in centroid with the original circle of radius R, the length of the arc described by its centre in moving from one extreme position to the other is

Sol. $CP = CQ - PQ = R - \frac{R}{8} = \frac{7R}{8}$ and $OP = OT + TQ = \frac{3R}{2} + \frac{R}{8} = \frac{13R}{8}$

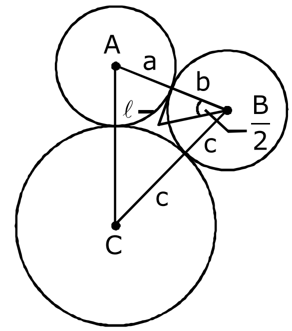


$$\text{Now } \cos 0 = \frac{R^2 + \left(\frac{7R}{8}\right)^2 - \left(\frac{13}{8}\right)^2 R^2}{2.R.\frac{7R}{8}}$$

$$= \frac{1 + \frac{49}{64} - \frac{169}{64}}{\frac{7}{4}} = -\frac{56}{112} = -\frac{1}{2}$$

$\Rightarrow$ $\theta = 120º \Rightarrow \alpha = 120º \Rightarrow PS = (CP)\,\alpha = \frac{7R}{8} . \frac{2\pi}{3} = \frac{7\pi R}{12}$

Ex.55 Three circles whose radii are a, b and c and c touch one other externally and the tangents at their points of contact meet in a point. Prove that the distance of this point from either of their points of contact is $\left(\dfrac{abc}{a+b+c}\right)^{1/2}$.

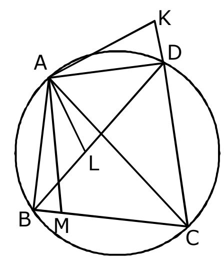


Sol. $\tan\dfrac{B}{2} = \dfrac{\ell}{b}$; $\tan\dfrac{A}{2} = \dfrac{\ell}{a}$ and $\tan\dfrac{C}{a} = \dfrac{\ell}{c}$.

Now $\Sigma \tan\dfrac{A}{2}\tan\dfrac{B}{2} = \dfrac{\ell^2}{abc}(a+b+c) \Rightarrow \ell^2 = \dfrac{abc}{a+b+c}$

Ex.56 Suppose ABCD is a cyclic quadrilateral and x, y, z are the distances A from the lins BD, BC, BD respectively. Prove that $\dfrac{BD}{x} = \dfrac{BC}{y} + \dfrac{CD}{z}$.

Sol. Let K, L M be the feet of perpendicular from A to CD, BD and BC respectively.
(Note that one foot is outside the circle in general.)
We have AL = x, AM = y, AK = z, let $\beta = \angle ADB = \angle ACB$, $\gamma = \angle ABC = \angle ADK$, $\delta = \angle ABD = \angle ACD$



Now $\dfrac{BC}{y} + \dfrac{CD}{z} = \dfrac{MB+MC}{y} + \dfrac{CK+DK}{z} = \dfrac{BM}{y} + \dfrac{MC}{y} + \dfrac{CK}{z} + \dfrac{DK}{z}$

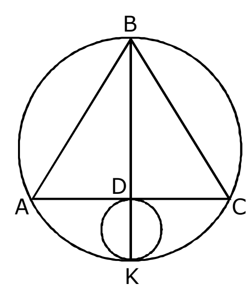
$= \cot\gamma + \cot\beta + \cot\delta - \cot\gamma = \cot\beta + \cot\delta$

$= \dfrac{DL}{x} + \dfrac{BL}{x}$ (from triangles ADL and ABL) $= \dfrac{BD}{x}$

Thus we have the desired relation.

Ex.57 Inscribed in a circle is an isosceles triangle ABC whose base AC = b and the base angle is α. A second circle touches the first circle and the base of the triangle at its midpoint D, and is situated outside the triangle. Find the radius of the second circle.

Sol. Let us take advantage of the fact that AD.DC = BD.DK

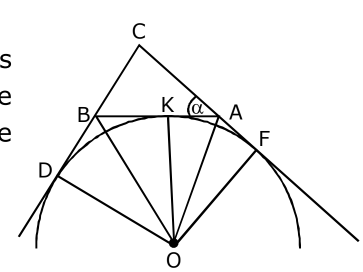


(Theorem 16a). Since AD = DC = $\dfrac{b}{2}$; BD = $\dfrac{b}{2}\tan\alpha$,

and DK = 2r, we get : $\dfrac{b^2}{4} = \dfrac{b}{2}\tan\alpha \cdot 2r$, hence $r = \dfrac{b}{4}\cot\alpha$.

Ex.58 An acute angle of a right triangle equals α. Find the hypotenuse of this triangle if the radius of the circle touching the hypotenuse and the extended legs is equal to R.

Sol. Since AB = AK + BK, the problem is reduced to computing the line segments AK (from ΔAOK) and BK {from ΔOBK). Consider the triangle AOK. Since $\angle KOF = \angle BAC = \alpha$ (as angles with mutually perpendicular sides), we have



$\angle KOA = \dfrac{\alpha}{2}$ ($\Delta KOA = \Delta AOF$). Hence, $AK = OK\tan\dfrac{\alpha}{2} = R\tan\dfrac{\alpha}{2}$.

Consider the triangle BOK. We have : $\angle BOK = \frac{1}{2}\angle DOK = \frac{1}{2}(90^\circ - \angle KOF) = 45^\circ - \frac{\alpha}{2}$. (Here we have taken advantage of the fact in the quadrilateral ODCF three angles (D, C, and F) are right and hence the fourth angle, that is the angle DOF, is also right) Then $BK = OK \tan \angle BOK = R \tan\left(45^\circ - \frac{\alpha}{2}\right)$,

$$AB = AK + BK = R\tan\frac{\alpha}{2} + R\tan\left(45^\circ - \frac{\alpha}{2}\right) = R\,\frac{\sin\left(\frac{\alpha}{2} + 45^\circ \frac{\alpha}{2}\right)}{\cos\frac{\alpha}{2}\cos\left(45^\circ \frac{\alpha}{2}\right)} = \frac{R\sqrt{2}}{2\cos\frac{\alpha}{2}\cos\left(45^\circ - \frac{\alpha}{2}\right)}.$$

(Here we have used the formulae $\tan\alpha + \tan\beta) = \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$.)

R. AMBIGUOUS CASE OF SOLUTION OF TRIANGLE

To solve a triangle having given two sides and an angle opposite to one of them.

Let a, b, A be given ; then B is to found from the equation $\sin B = \frac{b}{a}\sin A$.

(a) If $a < b \sin A$, then $\frac{b\sin A}{a} > 1$, so that $\sin B > 1$, which is impossible. Thus there is no solution.

(b) If $a = b \sin A$, then $\frac{b\sin A}{a} = 1$, so that $\sin B = 1$ and B has only the value 90º.

(c) If $a > b \sin A$, then $\frac{b\sin A}{a} < 1$, and two values for B may be found from $\sin B = \frac{b\sin A}{a}$.

These values are supplementary, so that one angle is acute, the other obtuse.

(1) If $a < b$, then $A < B$, and therefore B may either be acute obtuse, so that both values are admissible. This is known as the ambiguous case.

(2) If $a = b$, then $A = B$; and if $a > b$, then $A > B$; in either case B cannot be obtuse, and therefore only the smaller value of B is admissible. When B is found, C is determined from $C = 180^\circ - A - B$.

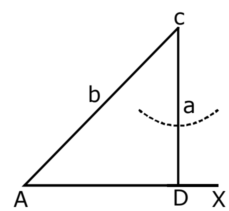
Finally, c may be found from the equation $c = \frac{a\sin C}{\sin A}$.

From the foregoing investigation it appears that the only case in which an ambiguous solution, can arise is when the smaller of the two given sides is opposite to the given angle.

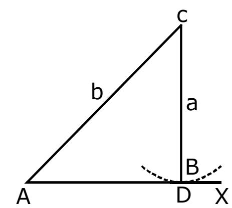
To discuss the Ambiguous case geometrically.

Let a, b, A be the given parts. Take a line AX unlimited towards X; mae $\angle$XAC equal to A, and AC equal to b. Draw CD perpendicular to AX, then CD = b sin A. With centre C and radius equal to describe a circle.

(a) If a < b sin A, the circle will not meet AX; thus no triangle can be constructed with the given parts.

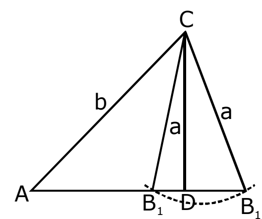


(b) If a = b sin A, the circle will touch AX at D; thus there is right-angled triangle with the given parts.

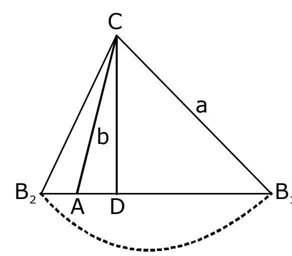


(c) If a > b sin A, the circle will cut AX in two points B_1, B_2.

(1) These points will be both on the same side of A, when a < b, in which case there are two solutions namely the triangles AB_1C, AB_2C This the Ambiguous case.

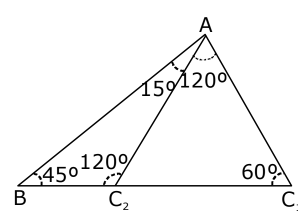


(2) The points B_1, B_2, will be on opposite sides of A when a >b. In this case there is only one solution, for the angle CAB_2 is the supplement of the given angle, and thus the triangle AB_2C does not satisfy the data.



(3) If a = b, the point B_2 coincides with A, so that there is only one solution.

Ex.59 Given B = 45º, c = $\sqrt{12}$, b = $\sqrt{8}$, solve the triangle.



Sol. We have $\sin C = \dfrac{c\sin B}{b} = \dfrac{2\sqrt{3}}{2\sqrt{2}} \cdot \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{3}}{2}$ $\quad\therefore\quad$ C = 60º or 120º

and since b < c, both these values are admissible.
The two triangles which satisfy the data are shown in the figure.
Denote the sides BC_1, BC_2 by a_1, a_2 and the angles BAC_1, BAC_2 by A_1, A_2 respectively

(a) In the ΔABC_1, $\angle A_1$ = 75º. Hence $a_1 = \dfrac{b\sin A_1}{\sin B} = \dfrac{2\sqrt{2}}{1} \cdot \dfrac{\sqrt{3}+1}{2\sqrt{2}} = \sqrt{2}\,(\sqrt{3} + 1)$

(b) In the ΔABC_2, $\angle A_2$ = 15º. Hence $a_2 = \dfrac{b\sin A_2}{\sin B} = \dfrac{2\sqrt{2}}{\frac{1}{\sqrt{2}}} \cdot \dfrac{\sqrt{3}-1}{2\sqrt{2}} = \sqrt{2}\,(\sqrt{3} - 1)$

Thus the complete solution is $\begin{cases} C = 60^o, \text{or } 120 \\ A = 75^o, \text{or } 15^o \\ a = \sqrt{6}+\sqrt{2}, \text{ or } \sqrt{6}-\sqrt{2} \end{cases}$

The ambiguous case may also be discussed by first finding the third side.

As before, let a, b, A be given, then cos A = $\dfrac{b^2+c^2-a^2}{2bc}$ $\therefore$ $c^2 - 2b\cos A \cdot c + b^2 - a^2 = 0$

By solving this quadratic equation in c, we obtain $c = b\cos A \pm \sqrt{b^2\cos^2 A + a^2 - b^2} = b\cos A \pm \sqrt{a^2 - b^2\sin^2 A}$

(a) When a < b sin A, the quantity under the radical is negative, and the values of c are impossible, so that there is no solution.

(b) When a = b sin A, the quantity under the radical is zero, and c= b cos A. Since sin A < 1, it follows that a < b, and therefore A < B. Hence the triangle is impossible unless the angle A is acute,m in which case c is positive and there is one solution.

(c) When a < b sin A, there are three cases to consider.

(1) Suppose a < b, then A < B, and as before the triangle is impossible unless A is acute.

In this case b cos A is positive. Also $\sqrt{a^2 - b^2\sin^2 A}$ is real and < $\sqrt{b^2 - b^2\sin^2 A}$

i.e. $\sqrt{a^2 - b^2\sin^2 A}$ < b cos A

Hence both values of c are real and positive, so that there are two solutions.

(2) Suppose a > b, then $\sqrt{a^2 - b^2\sin^2 A} > \sqrt{b^2 - b^2\sin^2 A}$ i.e. $\sqrt{a^2 - b^2\sin^2 A}$ > b cos A

Hence one value of c is positive and one value is negative, whether A is acute or obtuse, and in each case there is only one solution.

(3) Suppose a = b, then $\sqrt{a^2 - b^2\sin^2 A}$ = b cos A i.e. c = 2b cos A or 0;

hence there is only one solution when A is acute, and when A is obtuse the triangle is impossible.

Ex.60 If b, c, B are given, and if b < c, show that $(a_1 - a_2)^2 + (a_1 + a_2)^2 \tan^2 B = 4b^2$, where a_1, a_2 are the two values of the third side.

Sol. From the formula cos B = $\dfrac{c^2+a^2-b^2}{2ca}$. we have $a^2 - 2c\cos B \cdot a + c^2 - b^2 = 0$

But the roots of this equation are a_1 and a_2; hence by the theory of quadratic equations

$a_1 + a_2 = 2c\cos B$ and $a_1a_2 = c^2 - b^2$.

$\therefore$ $(a_1 - a_2) = (a_1 + a_2)^2 - 4a_1a_2 = 4c^2\cos^2 B - 4(c^2 - b^2)$

$\therefore$ $(a_1 - a_2)^2 + (a_1 + a_2)^2 \tan^2 B = 4c^2\cos^2 B - 4(c^2 - b^2) + 4c^2\cos^2 B \tan^2 B$

$= 4c^2(\cos^2 B + \sin^2 B) - 4c^2 + 4b^2 = 4b^2$